Regularity of Harmonic Maps between Alexandrov Spaces (II)

Xi-Ping Zhu

Sun Yat-sen University

July 13th, 2016

(Based on the joint work with Hui-Chun Zhang,
http://arxiv.org/abs/1311.1331)
Summer School in "Geometric Analysis on Riemannian and singular metric measure spaces", Como

Contents

- **1. Regularity of harmonic maps between singular spaces**

The regularity of harmonic maps into **singular spaces** was first studied by M. Gromov and R. Schoen in 1992.

Theorem (M. Gromov & R. Schoen)

Let u be an energy minimizing map from a smooth manifold to a complex X with non-positive curvature (NPC) in the sense of Alexandrov. Then it is a locally Lipschitz map.

1. Regularity of harmonic maps between singular spaces

After then, there have many researches on harmonic maps into or between singular spaces. For examples,

- harmonic maps from smooth Riemanian manifolds to NPC spaces, (N. Korevaar & R. Schoen (1993,1997))
- harmonic maps between Alexandrov spaces (F.-H. Lin (1997), J. Jost (1997))
- harmonic maps between metric spaces via a generalization of Dirichlet form, (J. Jost (1994,1997))
- harmonic maps between simplicial complexes, (J. Chen (1995))
- harmonic maps from a Riemannian polyhedron to an NPC space, (J. Eells & B. Fuglede (2001))
- harmonic maps between metric spaces via a method of semigroup, (K-T. Sturm (2005))
- harmonic maps from manifolds with singular Riemanian metric to smooth manifolds, (Y. G. Shi (1996), C. Y. Wang and others (2008,2013))

Theorem (J. Jost (1997), F. H. Lin (1997), independently)

Let Ω be a bounded domain in an Alexandrov space M (with curvature bounded below) and let Y be an Alexandrov space with non-positive curvature (NPC). Then any (energy minimizing) harmonic map u from Ω to Y is locally Hölder continuous in Ω.

More generally, the Hölder continuity also holds for harmonic maps $u : \Omega \subset X \to Y$ in the following caces:

X	Y	
simplicial complex	*NPC* complex	J. Chen (1995)
Riem. polyhedron	*NPC* space	Eells-Fuglede (2001)
manifold with L^∞-Riem. metric	*NPC* smooth	Y. G. Shi (1996) Ishizuka-Wang (2008)
metric space with a Dirichlet form	*NPC* smooth	J. Jost (1997)

Recall

- M. Gromov & R. Schoen (1992) proved the Lipschitz continuity of (energy minimizing) harmonic maps from a smooth Riemannian manifold to an NPC complex.

- N. Korevaar and R. Schoen (1993) extended the Lipschitz regularity as follows.

Theorem (N. Korevaar & R. Schoen)

Let Ω be a bounded domain of a smooth Riemannian manifold M, and Y be an Alexandrov space with non-positive curvature (NPC). Then any (energy minimizing) harmonic map $u : \Omega \to Y$ is locally Lipschitz continuous.

1. Regularity of harmonic maps between singular spaces

Based on the above theorem, F. H. Lin (1997) proposed an open problem:

Lin's Conjecture (F. H. Lin (1997))

Let Ω be a bounded domain of an Alexandrov space M and let Y be an Alexandrov space with non-positive curvature (NPC). Is an (energy minimizing) harmonic map u from Ω to Y locally Lipschitz continuous in Ω?

Remark: J. Jost (1998) also proposed a similar problem for Lipschitz regularity of harmonic maps between singular spaces.

1. Regularity of harmonic maps between singular spaces

- For maps between metric spaces, the best regularity one can expect is the Lipschitz continuity.
- The Lipschitz constant in the Korevaar-Schoen's theorem depends C^1-norm of metric coefficients g_{ij} on M,
- In 1996, J. Jost gave a different argument for the Korevaar-Schoen's Lipschitz regularity by using intersection properties of balls. The Lipschitz constant given by Jost depends on the upper and lower bounds of Ricci curvature on M.

To understand the difficulty, we can consider a simple case when $u \in W^{1,2}$ is a harmonic map from a domain $\Omega \subset \mathbb{R}^n$ with a singular metric $\{g_{ij}\}$ to a non-positively curved Riemannian manifold N^k (embedding into $\mathbb{R}^\ell$). Then $u \in W^{1,2}$ is a weak solution of elliptic system

$$\partial_i(\sqrt{g}g^{ij}\partial_j u_\alpha) + \sqrt{g}g^{ij}A_\alpha(\partial_i u, \partial_j u) = 0, \quad \alpha = 1, \cdots, \ell,$$

where $g = \det(g_{ij})$ and A is the second fundamental form of N^k in $\mathbb{R}^\ell$.

- If the coefficients $\sqrt{g}g^{ij}$ are merely in $L^\infty(\Omega)$, Y. G. Shi proved that u is Hölder continuous.

- But, even assume that all of coefficients $\sqrt{g}g^{ij}$ are continuous, u might fail to be Lipschitz continuous.

  Example:(T. Jin, V. Maz'ya & J. Schaftingen (2009))
  Let $\alpha(r) = \frac{-n}{(n-1)\log\frac{r_0}{r}}$ for some r_0 enough large, and let

  $$\sqrt{g}g^{ij} := a_{ij}(x) = \delta_{ij} + \alpha(|x|)(\delta_{ij} - \frac{x_i x_j}{|x|^2}).$$

  Then $a_{ij} \in C(B_o(1), \mathbb{R}^{n\times n})$ and $u = x_1 \cdot \log\frac{r_0}{|x|}$ solves

  $$\partial_i(a_{ij}\partial_j u) = 0$$

  on $B_o(1)$, but

  $$Du \notin L^\infty(B_o(1/2)).$$

Now let us consider M to be an Alexandrov spaces. Assume p is a regular point. It was showed by Y. Otsu and T. Shioya that there exists a coordinate neighborhood U around p and a BV_{loc}-metric $\{g_{ij}\}$ on U. We have

- $\{g_{ij}\}$ are in $L^{\infty}_{\mathrm{loc}}(U)$,
- but, it is well-known $\{g_{ij}\}$ may be NOT continuous on a *dense* subset of U (Y. Otsu and T. Shioya 1994).

Thus, it is apparent that the answer of the above Lin's conjecture might be negative.

On the other hand, let us consider the following example (due to J.-Y. Chen): the 2-dimensional metric cone with metric

$$dr^2 + a^2 r^2 d\theta^2$$

for some positive constant a. Then $u = r^{1/a} \cos\theta$ is harmonic.

- If $a \leqslant 1$, the metric cone is an Aexandrov space with nonnegative curvature and the harmonic function $u = r^{1/a} \cos\theta$ is Lipschitz continuous;
- If $a > 1$, the metric cone has curvature $-\infty$ at the vortex and the harmonic function $u = r^{1/a} \cos\theta$ is not Lipschitz continuous.

This example suggests Lin's conjecture might be correct!

- **2. Main theorem on Lipschitz continuity**

The above two examples show that the answer to Lin's conjecture should be subtle.

In November of 2013, H.-C. Zhang and I got an affirmative answer for the special case that the domain Alexandrov spaces have nonnegative curvature.

Finnaly, in December of 2014, we obtained a complete solution to Lin's conjecture

Theorem (H. C. Zhang & X. P. Zhu (2014))

Let Ω be a bounded domain in an n-dimensional Alexandrov space with curvature $\geqslant k$, and let (Y, d_Y) be an Alexandrov space with non-positive curvature (NPC). Then any (energy minimizing) harmonic map $u : \Omega \to Y$ is locally Lipschitz continuous. Moreover, we have the gradient estimate

$$\frac{d_Y\big(u(x),u(y)\big)}{|xy|} \leqslant C(n,k,R)\cdot\left(\left(\frac{E_2^u\big(B_q(R)\big)}{\mathrm{vol}\big(B_q(R)\big)}\right)^{1/2} + \mathrm{osc}_{\overline{B_q(R)}}u\right)$$

for all $x, y \in B_q(R/16)$.

PART 1. We introduce a family of (two-parameter) auxiliary functions $f_t(\cdot,\lambda)$ for $t,\lambda>0$, fixed $\Omega'\subset\subset\Omega$,

$$f_t(x,\lambda):=\inf_{y\in\Omega'}\left\{e^{-2nk\lambda}\frac{|xy|^2}{2t}-d_Y(u(x),u(y))\right\}.$$

We want to prove that there is a positive t_0 such that for each $t\in(0,t_0)$, the function $(x,\lambda)\mapsto f_t(x,\lambda)$ is a super-solution of the heat equation, i.e.

$$\frac{\partial f_t(x,\lambda)}{\partial\lambda}-\Delta f_t(x,\lambda)\geqslant 0$$

in the sense of measure.

Sketch of the proof–Part 1

Denote by

$$S_t(x,\lambda) := \Big\{ y \in \Omega' \mid f_t(x,\lambda) = e^{-2nk\lambda} \cdot \frac{|xy|^2}{2t} - d_Y\big(u(x), u(y)\big) \Big\},$$

and $\quad L_{t,\lambda}(x) := \mathrm{dist}\big(x, S_t(x,\lambda)\big) = \min_{y \in S_t(x,\lambda)} |xy|.$

Step 1. We first show

$$\frac{\partial f_t(x,\lambda)}{\partial \lambda} \geqslant -e^{-2nk\lambda} \cdot \frac{nk}{t} \cdot L^2_{t,\lambda}(x)$$

for $x \in \Omega'$.

Step 2. The **KEY** is to prove the following:

Given any point $p \in \Omega'$, there exits a neighborhood $U_p (= B_p(R_p))$ of p and a constant $t_p > 0$ such that, for each $t \in (0, t_p)$ and each $\lambda \in [0, 1]$, the function $x \mapsto f_t(x, \lambda)$ is a super-solution of the Poisson equation

$$\Delta f_t(x, \lambda) = -e^{-2nk\lambda} \cdot \frac{nk}{t} L^2_{t,\lambda}(x),$$

on U_p, in the sense of measure.

To illustrate the idea of the proof, I describe the sketch for the simple case that the domain Alexandrov spaces have nonnegative curvature (i.e. $k = 0$).
That is, we need to show the function

$$f_t(x, \lambda) = \inf_{y \in \Omega'} \left\{ e^{-2nk\lambda} \frac{|xy|^2}{2t} - d_Y(u(x), u(y)) \right\},$$

(with $k = 0$ and then independent of λ), is a super-solution of the Laplace equation

$$\Delta f_t(x) = 0$$

on U_p, in the sense of measure.

We argue by contradiction.

- Suppose not. Then $\exists\ \delta_0 > 0$, some sufficiently small $t > 0$ and some open set B such that $f_t + v$ has a strict minimum in B, where function v solves

$$\Delta v = -\delta_0 \ \text{ on } B, \qquad v + f_t = 0 \ \text{ on } \partial B.$$

- By the definition of f_t, the function

$$H(x, y) = \frac{|xy|^2}{2t} - d_Y(u(x), u(y)) + v(x)$$

has a minimum at $(\tilde{x}, \tilde{y})$ in $B \times \Omega$.

Substep 1. **To derive the mean value inequality for the function** $|xy|^2$.

- Denote by T the parallel translation of Petrunin (1998). For almost all points $(\bar{x}, \bar{y})$, there exists an isometry $T : T_{\bar{x}} \to T_{\bar{y}}$ such that we have the mean value inequality for function $|xy|^2$ around $(\bar{x}, \bar{y})$

$$\int_{B_o(\varepsilon_j)} \left(|\exp_{\bar{x}}(\eta)\ \exp_{\bar{y}}(T\eta)|^2 - |\bar{x}\bar{y}|^2 \right) dH^n(\eta) \leqslant o(\varepsilon_j^{n+2}).$$

  Here we used the fact that the domain Alexandrov spaces have nonnegative curvature.

- By a perturbation argument, we can assume that the minimum $(\tilde{x}, \tilde{y})$ of H satisfies the above inequality.

Substep 2. To derive the mean value inequality for the function $-d_Y(u(x), u(y))$.

(1) Considering the Lebesgue decomposition

$$E_2^u = |\nabla u|_2 \cdot \mathrm{vol} + (E_2^u)^s,$$

we have the following Calderon-Zygmund type differentiability property for Sobolev maps

Lemma

For any sequence $\{\epsilon_j\}_{j=1}^{\infty}$ with $\epsilon_j \to 0$, there exists a subsequence $\{\varepsilon_j\}_j \subset \{\epsilon_j\}_j$ such that, for almost everywhere $x_0 \in \Omega$, we have

$$\int_{B_{x_0}(\varepsilon_j)} d_Y^2(u(x), u(x_0))\, d\mathrm{vol}(x) = \frac{\omega_{n-1}}{n(n+2)} |\nabla u|_2(x_0) \cdot \varepsilon_j^{n+2} + o(\varepsilon_j^{n+2}).$$

(2) A mean value inequality for $d_Y^2(u(x), P)$, $\forall P \in Y$.

- Y is NPC $\Longrightarrow$ For any $P \in Y$, the function $d_Y^2(u(x), P)$ satisfies

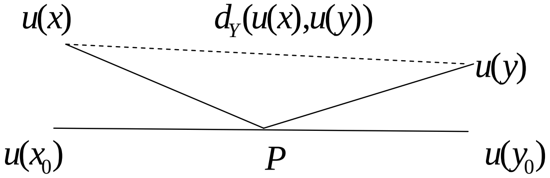
$$\Delta d_Y^2(u(x), P) \geqslant 2|\nabla u|_2 \cdot \mathrm{vol} \geqslant 2|\nabla d_Y(u(x), P)|^2 \cdot \mathrm{vol}.$$

- We have the mean value inequality for $d_Y^2(u(x), P)$, for almost everywhere $x_0 \in \Omega$,

$$\int_{B_{x_0}(\epsilon)} \Big[d_Y^2\big(u(x_0), P\big) - d_Y^2\big(u(x), P\big) \Big] d\mathrm{vol}(x)$$

$$\leqslant -\frac{|\nabla u|_2(x_0) \cdot \omega_{n-1}}{n(n+2)} \cdot \epsilon^{n+2} + o(\epsilon^{n+2}).$$

(3) A mean value inequality for $-d_Y(u(x), u(y))$.



$$\Big(d_Y\big(u(x_0), u(y_0)\big) - d_Y\big(u(x), u(y)\big)\Big) \cdot d_Y\big(u(x_0), u(y_0)\big)$$
$$\leqslant w_{x_0,P}(x) + w_{y_0,P}(y),$$

by Reshetnyak quadrilateral comparison (NPC of Y), where P is the mid-point and

$$w_{x_0,P}(\tau) := d_Y^2(u(\tau), u(x_0)) + d_Y^2(u(x_0), P) - d_Y^2(u(\tau), P).$$

Sketch of the proof–Part 1

- Using the differentiabilty property for $d_Y^2(u(x), u(x_0))$ and mean value inequality for $d_Y^2(u(x_0), P) - d_Y^2(u(x), P)$, we get for almost everywhere $x_0, y_0 \in \Omega$, and any isometry $T : T_{x_0} \to T_{y_0}$

$$\int_{B_o(\varepsilon_j)} \Big(d_Y(u(x_0), u(y_0)) - d_Y(u(\exp_{x_0}(\eta)), u(\exp_{y_0}(T\eta)) \Big) dH^n(\eta) \leqslant o(\varepsilon_j^{n+2})$$

——— the mean value inequality for $-d_Y(u(x), u(y))$.

- By a perturbation argument, we can also assume that the minimum $(\tilde{x}, \tilde{y})$ of H satisfies the above inequality.

Substep 3.

- Summary, the mean value of
$$H(x,y)=\frac{|xy|^2}{2t}-d_Y(u(x),u(y))+v(x),$$
around $(\tilde{x},\tilde{y})$ satisfies :
    - (since $(\tilde{x},\tilde{y})$ is a minimum), the mean value of $H(x,y)$ is $\geqslant 0$;
    - the mean value of both $|xy|^2$ and $-d_Y(u(x),u(y))$ are $\leqslant o(\varepsilon_j^{2+n})$;
    - (since $\Delta v=-\delta_0$), the mean value of v satisfies $\int_{B_o(\varepsilon_j)}[v(\exp_{\tilde{x}}(\eta))-v(\tilde{x})]dH^n(\eta)\leqslant -c_n\cdot\delta_0\cdot\varepsilon_j^{2+n}+o(\varepsilon_j^{2+n})$.
- A contradiction comes from the arbitrariness of ε_j, and then completes the Part 1.

PART 2. We set $v(t,x,\lambda) := -f_t(x,\lambda)$. Then v is nonnegative and a sub-solution of the heat equation .

- The key observation in this step is

$$\frac{\partial^+}{\partial t} v(t,x,\lambda) := \limsup_{s\to 0^+} \frac{v(t+s,x,\lambda)-v(t,x,\lambda)}{s}$$

$$\leqslant (\mathrm{Lip}u(x))^2 + |\nabla^+ v(t,x,\lambda)|^2, \qquad (*)$$

where

$$|\nabla^+ v(t,x,\lambda)| := \limsup_{y\to x} \frac{[v(t,y,\lambda)-v(t,x,\lambda)]_+}{|xy|}$$

and

$$\mathrm{Lip}u(x) := \limsup_{y\to x} \frac{d_Y(u(x),u(y))}{|xy|}.$$

Sketch of the proof–Part 2

- Since $v(t, x, \lambda)$ is bounded in $L^\infty(\Omega' \times (0,1))$ (uniformly in t) and is a sub-solution of the heat equation,

  $$\Longrightarrow |\nabla^+ v(t, x, \lambda)| \text{ is bounded in } L^2(\Omega'' \times (1/4, 3/4))$$

  (uniformly in t)

- $$+ \text{ the fact that } \mathrm{Lip} u \text{ is in } L^2(\Omega'')$$

  $$\overset{(*)}{\Longrightarrow} \frac{\partial^+}{\partial t} v(t, x, \lambda) \text{ is bounded in } L^1(\Omega'' \times (1/4, 3/4))$$

  (uniformly in t)

- $$+ \text{ the fact that } v(0, x, \lambda) = 0$$

  $$\Longrightarrow \int_{\Omega'' \times (1/4, 3/4)} v(t, x, \lambda) \leqslant Ct,$$

  where $\Omega'' \subset\subset \Omega'$

Sketch of the proof–Part 2

- Since $v(t,x,\lambda)/t$ is a sub-solution of the heat equation and

$$\int_{\Omega''\times(1/4,3/4)} v(t,x,\lambda) \leqslant Ct,$$

we have

$$\|\frac{v(t,x,1/2)}{t}\|_{L^\infty(\Omega''')} \leqslant C\int_{\Omega''\times(1/4,3/4)} \frac{v(t,x,\lambda)}{t} \leqslant C,$$

for all $t\in(0,t_0)$ with some fixed small $t_0>0$, where $\Omega'''\subset\subset\Omega''$.

Sketch of the proof–Part 2

- We then have

$$d_Y(u(x), u(y)) - \frac{e^{-nk}|xy|^2}{2t} \leqslant v(t, x, 1/2) \leqslant Ct,$$

for all $t \in (0, t_0)$ and all $x, y \in \Omega'''$.

- For any $|xy| < t_0$, choosing $t = |xy|$, we get

$$d_Y(u(x), u(y)) - \frac{e^{-nk}|xy|^2}{2|xy|} \leqslant C|xy|.$$

So, u is locally Lipschitz continuous.

Thank you for your attention !