Regularity of Harmonic Maps between Alexandrov Spaces (I)

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- **1. Introduction**

- **Energy**

Given a C^1-map $u:(M^n,g)\to(N^k,h)$ between two Riemannian manifolds of dimension n and k. Its energy is defined by

$$E(u):=\int_M e(u)d\mathrm{vol}_g,$$

where

$$e(u)(x):=\sum g^{\alpha\beta}(x)h_{ij}(u(x))\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}$$

and $\{x^\alpha\}_{1\leqslant\alpha\leqslant n}$, $\{u^i\}_{1\leqslant i\leqslant k}$ are local coordinates.

- **Harmonic map**

  A harmonic map is a critical point of $E(u)$.
  In particular, a (local) minimizer of $E(u)$ is called an **energy minimizing harmonic map**.

- If the target space $N = \mathbb{R}$, then the energy of u is

$$E(u) = \int_M |\nabla u|^2 d\mathrm{vol}_g$$

  and a harmonic map is a harmonic function on M.

Euler-Largange equation

From Nash embedding theorem, one can embed $N^k \hookrightarrow \mathbb{R}^\ell$. Now $E(u)$ can be rewritten as

$$E(u) = \sum_{i=1}^{\ell} \int_M |\nabla u^i|^2 d\mathrm{vol}_g.$$

A harmonic map $u \in W^{1,2}$ is a **weak** solution of the following elliptic system

$$\partial_\alpha(\sqrt{g} g^{\alpha\beta} \partial_\beta u^i) + \sqrt{g} g^{\alpha\beta} A^i(\partial_\alpha u, \partial_\beta u) = 0, \quad i = 1, \cdots, \ell,$$

where $g = \det(g_{\alpha\beta})$ and A is the second fundamental form of N^k in $\mathbb{R}^\ell$.

1. Introduction

The regularity of harmonic maps is one of the most important topics in geometric analysis.

- Every energy minimizing harmonic map is smooth provided $\dim_M = 2$.(C. B. Morrey (1948))
- If N has non-positive curvature, then there exists a smooth harmonic map (in each homotopy class). (J. Eells & J. H. Sampson (1964))
- If the image of energy minimizing harmonic map u is contained in a convex ball of N, it is smooth. (S. Hildebrandt, H. Kaul & K. O. Widman (1977))
- The general regularity theory has been given by R. Schoen & K. Uhlenbeck in 1982.

In particular, for the target manifold with non-positive sectional curvature, we have

Theorem (Hildebrandt-Kaul-Widman, Schoen-Uhlenbeck)

If the manifold N has nonpositive sectional curvature, then any energy minimizing harmonic map from M to N is smooth.

Remark: Without any restriction on N, an energy minimizing map might not be even continuous.

- **Sobolev spaces on singular spaces.**

Let (X, d) be a complete, separable and proper metric space, μ a Radon measure with $supp(\mu) = X$.

Let $\Omega \subset X$ be an open domain. Given $f \in C(\Omega)$ and $x \in \Omega$, define

$$\mathrm{Lip} f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x, y)}.$$

If x is isolated, we put $\mathrm{Lip} f(x) = 0$.

- For any $1 \leqslant p \leqslant \infty$ and $f \in Lip_{\mathrm{loc}}(\Omega)$, defined $W^{1,p}$**-norm**

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\mathrm{Lip} f\|_{L^p(\Omega)}.$$

- **Sobolov spaces.**

$$W^{1,p}(\Omega) := \text{the closure of } \{f \in Lip_{\rm loc}(\Omega) |\ \|f\|_{W^{1,p}} < \infty\}$$
$$\text{under the } W^{1,p} - \text{norm.}$$

- For a $W^{1,2}(\Omega)$-function, the **energy** (or **Dirichlet integral**) is

$$E(f) = \int_\Omega |{\rm Lip} f|^2 d\mu.$$

- **Energy and harmonic maps between singular spaces**.

M. Gromov and R. Schoen in 1992 initiated to study the theory of **harmonic maps between singular spaces**, motivated by the p-adic superrigidity for lattices in group of rank one.

We now discuss the definition of harmonic maps between **singular spaces**.

Consider a map $u : M \to Y$ between two general metric spaces (M, d_M) and (Y, d_Y). We say $u \in L^p(M, Y)$ if

- for any $Q \in Y$, the function $d_Y(Q, u(\cdot)) \in L^p(M)$.

1. Introduction

Consider a map $u \in L^p(M, Y)$, N. Korevaar and R. Schoen (1993) introduced the following:

- **approximating energy density** of u, for any $\epsilon > 0$,

$$e^u_{p,\epsilon}(x) := \frac{1}{\mu(B_x(\epsilon))} \int_{B_x(\epsilon)} \frac{d^p_Y(u(x), u(y))}{\epsilon^p} d\mu(y).$$

Remark: If Ω is a domain of $\mathbb{R}^n$ and $Y = \mathbb{R}$, then

$$e^u_{p,\epsilon}(x) = \frac{1}{\omega_n \epsilon^n} \int_{B_x(\epsilon)} \frac{|u(x+V) - u(x)|^p}{\epsilon^p} dV,$$

and, when $u \in C^1(\Omega)$,

$$\lim_{\epsilon \to 0^+} e^u_{p,\epsilon}(x) = c(n,p)|\nabla u(x)|^p.$$

- **approximating energy** of u, (as a functional on $C_0(M)$,)

$$E^u_{p,\epsilon}(\phi) := \int_M \phi(x) e^u_{p,\epsilon} d\mu(x)$$

for all $\phi \in C_0(M)$.

Theorem (N. Korevaar & R. Schoen 1993)

If Ω is a domain of a smooth Riemanian manifold, and Y is an arbitrary metric space, then

$$E^u_{p,\epsilon} \rightharpoonup E^u_p \quad \text{weakly}, \qquad (\text{as } \epsilon \to 0)$$

for some functional E^u_p on $C_0(\Omega)$.

The same convergence property

$$E^u_{p,\epsilon} \rightharpoonup E^u_p \quad \text{weakly}, \qquad (\text{as } \epsilon \to 0)$$

holds also for the following cases:

- If Ω is a domain of a Lipschitz manifold, (G. Gregori);
- If Ω is a domain of a Riemannian polyhedron and $p = 2$, (J. Eells & B. Fuglede);
- If Ω is a domain of an **Alexandrov space** with curvature bounded below (K. Kuwae & T. Shioya).

The limit functional E_p^u is called the **p-energy** (functional) of u.

Thus the **energy** E_2^u is well defined for a map u from an **Alexandrov space** to a **metric space**.

Remark: When M, Y are smooth manifolds, the energy E_2^u coincides with the original

$$E(u) = \int_M e(u) d\mathrm{vol}_g,$$

where $e(u)(x) := \sum g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}$.

A (local) minimizer of the energy E_2^u is called an **(energy minimizing) harmonic map**

- **2. Alexandrov spaces**

Let (X, d) be a complete metric space.

- **Length**: A Lipschitz curve $\gamma(t) : [a, b] \to X$, its Length is

$$L(\gamma) := \sup \left\{ \sum_{j=1}^{N} d(\gamma(t_j), \gamma(t_{j+1})) \middle| a = t_0 < t_1 < \cdots < t_{N+1} = b \right\},$$

where the "sup" is taken over all subdivisions of $[a, b]$;

- **Geodesic**: $\gamma(t) : [a, b] \to X$ is called a geodesic if

$$L(\gamma) = d(\gamma(a), \gamma(b)) \qquad \text{(with speed 1)};$$

- **Geodesic space**: (X, d) is called a geodesic space if every pair $p, q \in X$ can be connected by some geodesic;

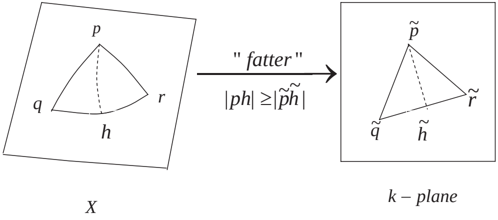
- **Triangle** $\triangle pqr$: a collection of three points p, q, r and three geodesics $[pq], [qr], [rp]$ in X;
- **k-plane**: 2-dim complete simply connected Riemannian manifolds of constant curvature k (i.e. $\mathbb{S}^2$, $\mathbb{R}^2$, $\mathbb{H}^2$);
- **Comparison triangle** $\widetilde{\triangle} pqr$ (of a triangle $\triangle pqr \subset X$): a triangle $\triangle \tilde{p}\tilde{q}\tilde{r}$ in a k-plane with

$$d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\tilde{q}, \tilde{r}) = d(q, r), \quad d(\tilde{r}, \tilde{p}) = d(r, p).$$

Definition: (A. Wald (40'), A. D. Alexandrov (40'–50'))

A geodesic space $(X, |\cdot\cdot|)$ is called an **Alexandrov space with curvature $\geqslant k$ (or $\leqslant k$ respectively)**, if

- any triangle $\triangle pqr$ in X is "fatter" (or "thiner" respectively) than the corresponding one in k-plane (for example, 0-plane is $\mathbb{R}^2$)



Examples for Alexandrov spaces of curvature bounded below:

- Riemannian manifolds with sec$\geqslant k$, Orbifolds with sec$\geqslant k$;
- The GH-limit spaces of manifolds (orbifolds) with sec$\geqslant k$ and dim $\leqslant n$;
- Convex polyhedrons in $\mathbb{R}^n$; Riemannian polyhedrons with cone-angle$\leqslant 2\pi$;
- Quotient spaces.
  If a group G acts isometrically on an Alexandrov space X, then X/G is also an Alexandrov space. (need not free!)

Basic properties: (Burago–Gromov–Perelman (1992))

Let X be an Alexandrov space with curvature bounded from below.

(i) The Hausdorff dimension is always an integer;

(ii) For any $\triangle pqr$ in X, the angle

$$\angle qpr \triangleq \lim_{x,y\to p} \angle \tilde{x}\tilde{p}\tilde{y}$$

is well defined and satisfies

$$\angle qpr \geqslant \angle \tilde{q}\tilde{p}\tilde{r};$$

Let $p \in X$, given two geodesic $\gamma(t)$ and $\sigma(s)$ with $\gamma(0) = \sigma(0) = p$, the angle

$$\angle(\gamma, \sigma) \triangleq \lim_{s,t\to 0} \angle\widetilde{\gamma(t)}\widetilde{p}\widetilde{\sigma(s)} \quad (\text{denote by} \quad \angle\gamma'(0)\sigma'(0))$$

is well defined.

- $\Sigma'_p(X)$= the space of equivalent classes of geodesics $\gamma(t)$ with $\gamma(0) = p$, where $\gamma(t) \sim \sigma(s)$ if $\angle(\gamma, \sigma) = 0$;
- $\Sigma_p(X)$= the metric completion of $\Sigma'_p(X)$ w.r.t. the angle metric.
  It is called the **space of directions** at p.

(iii) $\Sigma_p(X)$ is an Alexandrov space with curvature $\geqslant 1$ and $\dim(\Sigma_p(X)) = \dim(X) - 1$.

(iv) For any fixed $p \in X$, the Gromov-Hausdorff limit of dilation exists

$$\lim_{\lambda \to \infty} (X, p, \lambda d) = T_p(X),$$

where

- $T_p(X)$= the Euclidean cone over $\Sigma_p(X)$

$$= \left\{ (r, \xi) \middle| r \geqslant 0, \xi \in \Sigma_p(X) \right\} \text{ with the metric}$$

$$d^2_{T_p(X)}\big((r, \xi), (s, \eta)\big) \triangleq r^2 + s^2 - 2rs \cos \angle \xi\eta.$$

It is called the **tangent cone** at p.

On a tangent cone $T_p(X)$,

- $u\ (= (r, \xi_u) \in T_p(X))$ is called a **tangent vector** at p;
- $|u| \triangleq r$ and $|uv| = \sqrt{r^2 + s^2 - 2rs\cos\angle\xi_u\xi_v}$;
- **scalar product**, $\langle u, v\rangle \triangleq \frac{1}{2}(|u|^2 + |v|^2 - |uv|^2)$.

$\uparrow_p^x =$ the direction at p corresponding to *some* geodesic $[px]$, $x \neq p$.

$$W_p \triangleq \Big\{x \in X\backslash\{p\} \Big|\ \exists y (\neq x), \text{s.t. } d(p, y) = d(p, x) + d(x, y)\Big\}.$$

(v) W_p has full measure in X;
for each $x \in W_p$, the direction $\uparrow_p^x$ is *uniquely* determined.

Exponential map:

Let X be an $n-$dim Alexandrov space with curvature bounded from below.

Definitions: Fix any $p \in X$,

- $\log_p : W_p \to T_p(X)$; denote the image by $\mathscr{W}_p \triangleq \log_p(W_p)$.
- $\exp_p = (\log_p)^{-1} : \mathscr{W}_p \to W_p$.

Remark: Generally speaking, $\mathscr{W}_p$ may not contain any neighborhood of the vertex of $T_p(X)$.

Singularity: Let X be an n-dim Alexandrov space.

$p \in X$ is called **singular**, if its tangent cone $T_p(X) \neq \mathbb{R}^n$
(or **regular** if $T_p(X) = \mathbb{R}^n$, resp.).

Examples:

- a cube has singularity at each coner point;
- the cone over RP^2 has (topological) singularity at the vertex;
- the doubly-covered disc (all points are regular).

Remark:

(1) The singular set S_X is prossibly **dense** in X. (Otsu-Shioya);

(2) $\dim_H(S_X) \leqslant n-1$. (Burago–Gromov–Perelman).

Riemannian metric near a regular point: (Otsu-Shioya, Perelman)

Let p be regular point in X. Then

(1) $\exists$ a neighborhood U of p which is bi-Lipschitz onto an open domain in $\mathbb{R}^n$;

(2) One can define a ***BV*-Riemannian metric** $\{g_{\alpha\beta}\}$ on U s.t.,

- the distance function on U induced from $g_{\alpha\beta}$ coincides with the original one,
- the Riemannian measure on U coincides with the Hausdorff measure,
- $g_{\alpha\beta}$ is continuous in $U \backslash S_X$.

Remark:

The Riemannian metric $g_{\alpha\beta}$ might not be continuous on a dense set of U.

Parallel transportation (Petrunin, 1998)

Let X be an n-dim Alexandrov space with curvature $\geqslant k$. Assume $p, q \in X$ and the geodesic $[pq]$ can be extended beyond both p and q. Then,

- the tangent cone T_p is isometric to T_q;
- for any fixed sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$, $\epsilon_j \searrow 0$, $\exists$ $T: T_p \to T_q$ (an isometry preserving the vertexes) and a subsequence $\{\varepsilon_j\} \subset \{\epsilon_j\}$ such that

$$\big|\exp_p(\varepsilon_j\cdot\eta)\ \exp_q(\varepsilon_j\cdot T\eta)\big| \leqslant |pq| - \frac{k\cdot|pq|}{2}|\eta|^2\cdot\varepsilon_j^2 + o(\varepsilon_j^2)$$

for any $\eta \in T_p$ such that the left-hand side is well-defined.

Remark: Such an isometry T depends on the choices of sequences $\{\epsilon_j\}$ and $\{\varepsilon_j\}$.

Topology of Alexandrov spaces (of curvature bounded below)

- **Stability theorem** (Perelman 1991) :

$$\left.\begin{array}{l} X_i \xrightarrow{GH} X_\infty, \ \ as\ \ i \to +\infty, \\ \dim X_i = \dim X_\infty, \\ X_\infty \ \ compact \end{array}\right\} \Longrightarrow X_i \overset{homeo}{\cong} X_\infty, \ \ as\ \ i\ \ large$$

Consequently, an n-dimensional Alexandrov space with topological singularity (e.g. the cone over RP^2) can not be approximated by n-dimensional Riemannian manifolds.

Boundary of Alexandrov spaces

Definition: Let X be an n-dim Alexandrov space. Its **boundary** ∂X is defined inductively w.r.t. dimension n.

- If $\dim(X) = 1$, then X is a complete Riemannian manifold and ∂X is defined as usual;
- If $\dim(X) = n \geqslant 2$, a point $p \in \partial X$ if the space of directions $\Sigma_p(X)$ has non-empty boundary.

Remark:
(1) If $\partial X = \emptyset$, then the singular set S_X has $\dim_H(S_X) \leqslant n - 2$;

(2) In these two talks, we always consider Alexandrov spaces without boundary.

Thank you for your attention !