The splitting theorem in nonsmooth setting

Nicola Gigli, SISSA

July 15, 2016

Plan of the lectures

PART I: Introduction: the Cheeger-Gromoll splitting theorem and the definition of RCD spaces

PART II: Analysis on RCD spaces

PART III: Proof of the splitting in the non-smooth setting

PART I: Introduction

- The Cheeger-Gromoll splitting theorem
  - The original argument
  - Splitting without the Hessian
- The Riemannian curvature dimension condition
  - Metric measure spaces and their convergence
  - Optimal transport and the curvature-dimension condition
  - Functional analysis and the Riemannian curvature dimension condition

PART I: Introduction

- The Cheeger-Gromoll splitting theorem
  - The original argument
  - Splitting without the Hessian
- The Riemannian curvature dimension condition
  - Metric measure spaces and their convergence
  - Optimal transport and the curvature-dimension condition
  - Functional analysis and the Riemannian curvature dimension condition

A bit of history

The splitting theorem is a rigidity statement in Riemannian geometry.

A non-exhaustive account of its history:

- Cohn-Vossen '36 in $d = 2$
- Toponogov '59 for any d under the assumption $\mathrm{Sect} \geq 0$
- Cheeger-Gromoll '71 for any d under the assumption $\mathrm{Ric} \geq 0$

A bit of history

The splitting theorem is a rigidity statement in Riemannian geometry.

A non-exhaustive account of its history:

- Cohn-Vossen '36 in $d = 2$
- Toponogov '59 for any d under the assumption $\mathrm{Sect} \geq 0$
- Cheeger-Gromoll '71 for any d under the assumption $\mathrm{Ric} \geq 0$

Milka '67 for Alexandrov spaces, Newman '90 for Lorentzian manifolds, Borzellino-Zhu '94 for orbifolds

The splitting theorem

Thm. (Cheeger-Gromoll ’71)
Let M be a Riemannian manifold with $\mathrm{Ric} \geq 0$ which contains a line.

The splitting theorem

Thm. (Cheeger-Gromoll '71)
Let M be a Riemannian manifold with $\mathrm{Ric} \geq 0$ which contains a line.

Then $M = N \times \mathbb{R}$ for some Riemannian manifold N with $\mathrm{Ric} \geq 0$.

The Busemann function

Let $\gamma : [0, \infty) \to M$ an half line.
The Busemann function b associated to it is

$$\mathrm{b}(x) := \lim_{t \to +\infty} t - \mathrm{d}(x, \gamma_t) = \sup_{t \geq 0} t - \mathrm{d}(x, \gamma_t)$$

The Busemann function

Let $\gamma : [0, \infty) \to M$ an half line.
The Busemann function b associated to it is

$$\mathrm{b}(x) := \lim_{t \to +\infty} t - \mathrm{d}(x, \gamma_t) = \sup_{t \geq 0} t - \mathrm{d}(x, \gamma_t)$$

If $\gamma : (-\infty, +\infty) \to M$ is a line we can associate to it 2 Busemann functions

$$\mathrm{b}^+(x) := \lim_{t \to +\infty} t - \mathrm{d}(x, \gamma_t)$$
$$\mathrm{b}^-(x) := \lim_{t \to +\infty} t - \mathrm{d}(x, \gamma_{-t})$$

Laplacian comparison estimate and its use

If $\mathrm{Ric} \geq 0$ and $\bar{x} \in M$

$$\Delta \frac{\mathsf{d}^2(\cdot, \bar{x})}{2} \leq \dim(M)$$

Laplacian comparison estimate and its use

If $\mathrm{Ric} \geq 0$ and $\bar{x} \in M$

$$\Delta \frac{\mathsf{d}^2(\cdot, \bar{x})}{2} \leq \mathrm{dim}(M)$$

Hence

$$\Delta \mathsf{d}(\cdot, \gamma_t) \leq \frac{\mathrm{dim}(M)}{\mathsf{d}(\cdot, \gamma_t)}$$

Passing to the limit we obtain

$$\Delta \mathrm{b} \geq 0,$$

i.e. the b is subharmonic.

Use of the strong maximum principle

b^+ and b^- are subharmonic, thus so is $\mathrm{b}^+ + \mathrm{b}^-$.
The triangle inequality gives

$$\mathrm{b}^+ + \mathrm{b}^- \leq 0$$

and the fact that γ is a line ensures that

$$(\mathrm{b}^+ + \mathrm{b}^-)(\gamma_0) = 0$$

hence by the strong maximum principle it holds

$$\mathrm{b}^+ + \mathrm{b}^- \equiv 0$$

and in particular b^+ and b^- are harmonic

Use of the Bochner equality and inequality

For any f smooth it holds

$$\begin{aligned}\Delta \frac{|\nabla f|^2}{2} &= \|\mathrm{Hess}\, f\|^2_{\mathrm{HS}} + \nabla f \cdot \nabla \Delta f + \mathrm{Ric}(\nabla f, \nabla f)\\ &\geq \frac{(\Delta f)^2}{\dim(M)} + \nabla f \cdot \nabla \Delta f\end{aligned}$$

Use of the Bochner equality and inequality

For any f smooth it holds

$$\Delta \frac{|\nabla f|^2}{2} = \|\mathrm{Hess}\, f\|_{\mathrm{HS}}^2 + \nabla f \cdot \nabla \Delta f + \mathrm{Ric}(\nabla f, \nabla f)$$
$$\geq \frac{(\Delta f)^2}{\dim(M)} + \nabla f \cdot \nabla \Delta f$$

For b^+ we have $|\nabla \mathrm{b}^+| \equiv 1$ and $\Delta \mathrm{b}^+ \equiv 0$ and thus the equality

$$\Delta \frac{|\nabla \mathrm{b}^+|^2}{2} = \frac{(\Delta \mathrm{b}^+)^2}{\dim(M)} + \nabla \mathrm{b}^+ \cdot \nabla \Delta \mathrm{b}^+$$

Use of the Bochner equality and inequality

For any f smooth it holds

$$\Delta\frac{|\nabla f|^2}{2} = \|\mathrm{Hess}\, f\|^2_{\mathrm{HS}} + \nabla f\cdot\nabla\Delta f + \mathrm{Ric}(\nabla f,\nabla f)$$
$$\geq \frac{(\Delta f)^2}{\dim(M)} + \nabla f\cdot\nabla\Delta f$$

For b^+ we have $|\nabla \mathrm{b}^+|\equiv 1$ and $\Delta\mathrm{b}^+\equiv 0$ and thus the equality

$$\Delta\frac{|\nabla \mathrm{b}^+|^2}{2} = \frac{(\Delta\mathrm{b}^+)^2}{\dim(M)} + \nabla\mathrm{b}^+\cdot\nabla\Delta\mathrm{b}^+$$

which yields

$$\|\mathrm{Hess}\,\mathrm{b}^+\|^2_{\mathrm{HS}} \equiv \frac{(\Delta\mathrm{b}^+)^2}{\dim(M)} \equiv 0$$

i.e. b^+ is both convex and concave.

Isometries via gradient flows

Since b^+ is convex, its gradient flow contracts distances.

Since $\mathrm{b}^+ = -\mathrm{b}^-$ is concave, its gradient flow expands distances.

Thus the gradient flow (F_t) of b^+ produces a 1-parameter family of isometries.

Conclusion of the argument

Put $N := \{\mathrm{b}^+ = 0\}$ and define $\Phi : N \times \mathbb{R} \to M$ as $\Phi(x, t) := F_t(x)$.

Φ is clearly smooth and a bijection.

It is clear that $\mathrm{d}\Phi : T_{(x,t)} N \times \mathbb{R} \to T_{\Phi(x,t)} M$ is and isometry when restricted to $T_x N \times \{0\}$ and $\{0\} \times T_t \mathbb{R}$

It remains to show that $\mathrm{d}\Phi$ sends a vector in $T_x N \times \{0\}$ and one in $\{0\} \times T_t \mathbb{R}$ to orthogonal vectors, but this is trivial because $\nabla \mathrm{b}^+$ is orthogonal to $T_x N \subset T_x M$.

PART I: Introduction

- The Cheeger-Gromoll splitting theorem
  - The original argument
  - Splitting without the Hessian
- The Riemannian curvature dimension condition
  - Metric measure spaces and their convergence
  - Optimal transport and the curvature-dimension condition
  - Functional analysis and the Riemannian curvature dimension condition

What we aim to do

Say that we already proved that the Busemann function b is harmonic.

We want to conclude without calling into play the Hessian and relying only on

$$\Delta \frac{|\nabla f|^2}{2} \geq \langle \nabla f, \nabla \Delta f \rangle$$

Notice that as before we have

$$\Delta \frac{|\nabla \mathrm{b}|^2}{2} = \langle \nabla \mathrm{b}, \nabla \Delta \mathrm{b} \rangle$$

‘Euler equation’ for b

Write the Bochner inequality for $\mathrm{b} + \varepsilon f$ to get

$$\Delta \frac{|\nabla(\mathrm{b} + \varepsilon f)|^2}{2} \geq \langle \nabla(\mathrm{b} + \varepsilon f), \nabla \Delta(\mathrm{b} + \varepsilon f) \rangle$$

Expanding we obtain

$$\varepsilon \Delta(\nabla \mathrm{b} \cdot \nabla f) \geq \varepsilon \nabla \mathrm{b} \cdot \nabla \Delta f + O(\varepsilon^2)$$

and thus

$$\Delta(\nabla \mathrm{b} \cdot \nabla f) = \nabla \mathrm{b} \cdot \nabla \Delta f$$

The gradient flow of b preserves the volume measure

Let (F_t) be the gradient flow of b, i.e.

$$\partial_t F_t(x) = \nabla \mathrm{b}(F_t(x)) \qquad F_0(x) = x.$$

We claim that $(F_t)_* \mathrm{vol} = \mathrm{vol}$.
Indeed, for f smooth, putting $f_t := f \circ F_t$ we have

$$\partial_t f_t = \lim_{h \to 0} \frac{f_t \circ F_h - f_t}{h} = \nabla f_t \cdot \nabla \mathrm{b}$$

and thus

$$\begin{aligned} \partial_t \int f \, \mathrm{d}(F_t)_* \mathrm{vol} &= \partial_t \int f \circ F_t \, \mathrm{dvol} \\ &= \int \nabla f_t \cdot \nabla \mathrm{b} \, \mathrm{dvol} \\ &= -\int f_t \Delta \mathrm{b} \, \mathrm{dvol} = 0 \end{aligned}$$

Use of the Euler equation

Let's compute

$$\partial_t \frac{1}{2} \int |\nabla f_t|^2 \, \mathrm{dvol} = \int \nabla f_t \cdot \nabla(\partial_t f_t) = \int \nabla f_t \cdot \nabla(\nabla f_t \cdot \nabla \mathrm{b}) \mathrm{dvol}$$

From the Euler equation we obtain

$$\begin{aligned} \int \nabla f_t \cdot \nabla(\nabla f_t \cdot \nabla \mathrm{b}) \mathrm{dvol} &= -\int f_t \Delta(\nabla f_t \cdot \nabla \mathrm{b}) \mathrm{dvol} \\ &= -\int f_t \nabla \mathrm{b} \cdot \nabla \Delta f_t \mathrm{dvol} \\ &= \int \nabla f_t \cdot \nabla \mathrm{b} \, \Delta f_t \mathrm{dvol} \\ &= -\int \nabla(\nabla f_t \cdot \nabla \mathrm{b}) \cdot \nabla f_t \mathrm{dvol} \end{aligned}$$

thus

$$\frac{1}{2} \int |\nabla f_t|^2 \, \mathrm{dvol} = \frac{1}{2} \int |\nabla f|^2 \, \mathrm{dvol}$$

Localizing the information

By polarization we get

$$\int \langle \nabla f_t, \nabla g_t \rangle \,\mathrm{dvol} = \int \langle \nabla f, \nabla g \rangle \,\mathrm{dvol}$$

thus

$$\begin{aligned}
\int g_t |\nabla f_t|^2 \,\mathrm{dvol} &= \int \langle \nabla (g_t f_t), \nabla f_t \rangle - \tfrac{1}{2} \langle \nabla g_t, \nabla (f_t^2) \rangle \,\mathrm{dvol} \\
&= \int \langle \nabla (g f), \nabla f \rangle - \tfrac{1}{2} \langle \nabla g, \nabla (f^2) \rangle \,\mathrm{dvol} \\
&= \int g |\nabla f|^2 \,\mathrm{dvol} = \int g_t |\nabla f|^2 \circ F_t \,\mathrm{dvol}
\end{aligned}$$

and thus

$$|\nabla f| \circ F_t = |\nabla (f \circ F_t)|$$

Localizing the information

By polarization we get

$$\int \langle \nabla f_t, \nabla g_t \rangle \,\mathrm{dvol} = \int \langle \nabla f, \nabla g \rangle \,\mathrm{dvol}$$

thus

$$\begin{aligned}\int g_t |\nabla f_t|^2 \,\mathrm{dvol} &= \int \langle \nabla (g_t f_t), \nabla f_t \rangle - \tfrac{1}{2} \langle \nabla g_t, \nabla (f_t^2) \rangle \,\mathrm{dvol} \\ &= \int \langle \nabla (g f), \nabla f \rangle - \tfrac{1}{2} \langle \nabla g, \nabla (f^2) \rangle \,\mathrm{dvol} \\ &= \int g |\nabla f|^2 \,\mathrm{dvol} = \int g_t |\nabla f|^2 \circ F_t \,\mathrm{dvol}\end{aligned}$$

and thus

$$|\nabla f| \circ F_t = |\nabla (f \circ F_t)| = |\mathrm{d}F_t(\nabla f)|$$

i.e. $\mathrm{d}F_t$ is an isometry.

PART I: Introduction

- The Cheeger-Gromoll splitting theorem
  - The original argument
  - Splitting without the Hessian
- The Riemannian curvature dimension condition
  - Metric measure spaces and their convergence
  - Optimal transport and the curvature-dimension condition
  - Functional analysis and the Riemannian curvature dimension condition

Some terminology and basic assumptions

Metric spaces will always be complete and separable.
Given such space (X, d), $\mathscr{P}(X)$ is the space of Borel probability measures on it and

$$\mathscr{P}_2(X) := \left\{ \mu \in \mathscr{P}(X) : \int \mathsf{d}^2(\cdot, \bar{x}) \, \mathrm{d}\mu < \infty \quad \text{for some, and thus all, } \bar{x} \in X \right\}$$

Given $T : X \to Y$ Borel and $\mu \in \mathscr{P}(X)$, the measure $T_*\mu$ on Y is defined as

$$T_*\mu(E) := \mu(T^{-1}(E)) \qquad \forall E \subset Y \text{ Borel}$$

and is characterized by

$$\int f \, \mathrm{d}T_*\mu = \int f \circ T \, \mathrm{d}\mu$$

Continuous curves

We shall denote by $C([0,1],X)$ the space of continuous curves on X equipped with the 'sup' distance.

It is complete and separable if (X,d) is.

For $t \in [0,1]$ the evaluation map $\mathrm{e}_t : C([0,1],X) \to X$ is defined by

$$\mathrm{e}_t(\gamma) := \gamma_t$$

Absolutely continuous curves

$\gamma \in C([0,1],X)$ is said **absolutely continuous** provided there exists $f \in L^1(0,1)$ such that

$$\mathsf{d}(\gamma_t,\gamma_s) \le \int_t^s f(r)\,\mathrm{d}r \qquad \forall t < s,\ t,s \in [0,1] \tag{1}$$

For such curve the limit

$$|\dot\gamma_t| := \lim_{h\to 0} \frac{\mathsf{d}(\gamma_{t+h},\gamma_t)}{|h|}$$

exists for a.e. $t \in [0,1]$ and is the a.e. minimal function f which can be chosen in (1)

Absolutely continuous curves

$\gamma \in C([0,1],X)$ is said **absolutely continuous** provided there exists $f \in L^1(0,1)$ such that

$$\mathrm{d}(\gamma_t, \gamma_s) \le \int_t^s f(r)\,\mathrm{d}r \qquad \forall t < s,\ t,s \in [0,1] \tag{1}$$

For such curve the limit

$$|\dot\gamma_t| := \lim_{h\to 0} \frac{\mathrm{d}(\gamma_{t+h}, \gamma_t)}{|h|}$$

exists for a.e. $t \in [0,1]$ and is the a.e. minimal function f which can be chosen in (1)
Idea of the proof: prove that the limit is equal to

$$\sup_n g_n' \qquad \text{where} \qquad g_n(t) := \mathrm{d}(\gamma_t, x_n)$$

and $(x_n) \subset X$ is dense

Length and geodesic spaces

(X, d) is a **length space** provided

$$\mathrm{d}(x, y) = \inf \int_0^1 |\dot{\gamma}_t| \, \mathrm{d}t, \tag{2}$$

the inf being taken among all absolutely continuous γ's with $\gamma_0 = x$, $\gamma_1 = y$.
(X, d) is a **geodesic space** if the inf is a min.

Length and geodesic spaces

(X, d) is a **length space** provided

$$\mathrm{d}(x, y) = \inf \int_0^1 |\dot{\gamma}_t| \, \mathrm{d}t, \qquad (2)$$

the inf being taken among all absolutely continuous γ's with $\gamma_0 = x$, $\gamma_1 = y$.
(X, d) is a **geodesic space** if the inf is a min.
A **geodesic** is a curve γ such that

$$\mathrm{d}(\gamma_t, \gamma_s) = |s - t| \mathrm{d}(\gamma_0, \gamma_1) \qquad \forall t, s \in [0, 1]$$

i.e. a minimizer of (2) parametrized with constant speed.
$\mathrm{Geo}(X) \subset C([0, 1], X)$ is the space of all geodesics on X.

Geodesic convexity

A functional $E : X \to \mathbb{R} \cup \{+\infty\}$ is said K**-convex** for $K \in \mathbb{R}$ provided for every $x, y \in X$ with $E(x), E(y) < \infty$ there exists a geodesic γ connecting them such that

$$E(\gamma_t) \leq (1-t)E(x) + tE(y) - \frac{K}{2}t(1-t)\mathrm{d}^2(x,y)$$

Metric measure spaces

For us, a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is a (complete and separable) metric space equipped with a non-negative Borel measure $\mathfrak{m}$ which is finite on bounded sets.

Metric measure spaces

For us, a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is a (complete and separable) metric space equipped with a non-negative Borel measure $\mathfrak{m}$ which is finite on bounded sets.

Two metric measure spaces $(X_1, \mathsf{d}_1, \mathfrak{m}_1), (X_2, \mathsf{d}_2, \mathfrak{m}_2)$ are isomorphic provided there is a measure preserving isometry from $(\mathrm{supp}(\mathfrak{m}_1), \mathsf{d}_1, \mathfrak{m}_1)$ to $(X_2, \mathsf{d}_2, \mathfrak{m}_2)$.
In particular, $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to $(\mathrm{supp}(\mathfrak{m}), \mathsf{d}, \mathfrak{m})$

Weak convergence of measures

Let ν_n, ν be non-negative Borel measures on X finite on bounded sets.
We say that (ν_n) weakly converges to ν and write $\nu_n \rightharpoonup \nu$ provided

$$\int \varphi \, \mathrm{d}\nu_n \qquad \to \qquad \int \varphi \, \mathrm{d}\nu$$

for every $\varphi \in C_b(X)$ with bounded support.

pointed-measured-Gromov-Hausdorff convergence of metric measure spaces

We say that $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ converges to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ provided there exists a metric space (Y, d_Y) and isometric embeddings ι_n, ι_∞ of $(X_n, \mathsf{d}_n), (X_\infty, \mathsf{d}_\infty)$ in (Y, d_Y) such that

$$\iota_n(\bar{x}_n) \to \iota_\infty(\bar{x}_\infty)$$

$$(\iota_n)_* \mathfrak{m}_n \rightharpoonup (\iota_\infty)_* \mathfrak{m}_\infty$$

Gromov's precompactness theorem

Theorem (Gromov '80) Let $(M_n, \bar{x}_n)$ be a sequence of pointed Riemannian manifolds equipped with the distance d_n induced by the metric tensor and the measure $\mathfrak{m}_n := c_n \mathrm{vol}_n$, where $c_n > 0$ is so that

$$c_n \, \mathrm{vol}_n(B_1(\bar{x}_n) = 1.$$

Assume that for some $K \in \mathbb{R}$, $N \geq 1$ it holds $\mathrm{Ric}_n \geq K$ and $\dim(M_n) \leq N$.

Then the sequence is precompact in the pmGH-convergence.

Gromov's precompactness theorem

Theorem (Gromov '80) Let $(M_n, \bar{x}_n)$ be a sequence of pointed Riemannian manifolds equipped with the distance d_n induced by the metric tensor and the measure $\mathfrak{m}_n := c_n \mathrm{vol}_n$, where $c_n > 0$ is so that

$$c_n \, \mathrm{vol}_n(B_1(\bar{x}_n)) = 1.$$

Assume that for some $K \in \mathbb{R}$, $N \geq 1$ it holds $\mathrm{Ric}_n \geq K$ and $\dim(M_n) \leq N$.

Then the sequence is precompact in the pmGH-convergence.

Idea of the proof: The Bishop-Gromov inequality grants that

$$\mathfrak{m}_n(B_{2r}(x)) \leq C(N, K)\mathfrak{m}_n(B_r(x)) \qquad \forall n \in \mathbb{N}, x \in M_n,\ r > 0.$$

This yields a uniform metric doubling of the (M_n, d_n)'s and thus uniform compactness.

The almost splitting theorem

Theorem (Cheeger-Colding '96) Let M be a Riemannian manifold with $\mathrm{Ric} \geq -\varepsilon$ and containing a geodesic of length $\geq \frac{1}{\varepsilon}$ whose midpoint we call $\bar{x}$.
Then $(M, \bar{x})$ is $f(\varepsilon, \dim(M))$-close in the pmGH topology to the product of a geodesic metric measure space M' and the real line for some $f : (0, \infty) \times \mathbb{R}^+$ such that

$$\lim_{\varepsilon \downarrow 0} f(\varepsilon, N) = 0 \qquad \forall N \in \mathbb{R}^+.$$

The almost splitting theorem

Theorem (Cheeger-Colding '96) Let M be a Riemannian manifold with $\mathrm{Ric} \geq -\varepsilon$ and containing a geodesic of length $\geq \frac{1}{\varepsilon}$ whose midpoint we call $\bar{x}$.
Then $(M, \bar{x})$ is $f(\varepsilon, \dim(M))$-close in the pmGH topology to the product of a geodesic metric measure space M' and the real line for some $f : (0, \infty) \times \mathbb{R}^+$ such that

$$\lim_{\varepsilon \downarrow 0} f(\varepsilon, N) = 0 \qquad \forall N \in \mathbb{R}^+.$$

Corollary (Splitting for limit spaces)
Let $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ be the pmGH-limit of a sequence of manifolds M_n with $\mathrm{Ric}(M_n) \geq -\varepsilon_n$, $\varepsilon_n \downarrow 0$, and $\sup \dim(M_n) < \infty$.
Assume that X contains a line. Then $X \sim Y \times \mathbb{R}$ for some geodesic mm-space Y, the product distance being

$$\mathsf{d}^2_{Y \times \mathbb{R}}\big((y_1, t_1), (y_2, t_2)\big) := \mathsf{d}^2_Y(y_1, y_2) + |t_1 - t_2|^2$$

The almost splitting theorem

Theorem (Cheeger-Colding '96) Let M be a Riemannian manifold with $\mathrm{Ric} \geq -\varepsilon$ and containing a geodesic of length $\geq \frac{1}{\varepsilon}$ whose midpoint we call $\bar{x}$.
Then $(M, \bar{x})$ is $f(\varepsilon, \dim(M))$-close in the pmGH topology to the product of a geodesic metric measure space M' and the real line for some $f : (0, \infty) \times \mathbb{R}^+$ such that

$$\lim_{\varepsilon \downarrow 0} f(\varepsilon, N) = 0 \qquad \forall N \in \mathbb{R}^+.$$

Corollary (Splitting for limit spaces)
Let $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ be the pmGH-limit of a sequence of manifolds M_n with $\mathrm{Ric}(M_n) \geq -\varepsilon_n$, $\varepsilon_n \downarrow 0$, and $\sup \dim(M_n) < \infty$.
Assume that X contains a line. Then $X \sim Y \times \mathbb{R}$ for some geodesic mm-space Y, the product distance being

$$\mathsf{d}^2_{Y \times \mathbb{R}}\big((y_1, t_1), (y_2, t_2)\big) := \mathsf{d}^2_Y(y_1, y_2) + |t_1 - t_2|^2$$

Important remark: by Gromov's theorem, also the converse implication holds.

PART I: Introduction

- The Cheeger-Gromoll splitting theorem
  - The original argument
  - Splitting without the Hessian
- The Riemannian curvature dimension condition
  - Metric measure spaces and their convergence
  - Optimal transport and the curvature-dimension condition
  - Functional analysis and the Riemannian curvature dimension condition

Transport plans

Given $\mu, \nu \in \mathscr{P}(X)$ we say that $\boldsymbol{\alpha} \in \mathscr{P}(X^2)$ is an admissible transport plan, and write $\boldsymbol{\alpha} \in \mathscr{Adm}(\mu, \nu)$ provided

$$\pi^1_* \boldsymbol{\alpha} = \mu \qquad \pi^2_* \boldsymbol{\alpha} = \nu,$$

where $\pi^i : X^2 \to X$ are the projection on the coordinates.
Every map T such that $T_* \mu = \nu$ induces a transport plan via the formula

$$\boldsymbol{\alpha}_T = (Id, T)_* \mu$$

The quadratic Kantorovich distance

For $\mu, \nu \in \mathscr{P}_2(X)$ we put

$$\frac{1}{2} W_2^2(\mu,\nu) := \min_{\boldsymbol{\alpha} \in \mathcal{Adm}(\mu,\nu)} \frac{1}{2} \int \mathsf{d}^2(x,y)\,\mathrm{d}\boldsymbol{\alpha}(x,y)$$

The minimum is always realized, a plan realizing the minimum will be called **optimal** and the set of optimal plans is denoted by $\mathcal{Opt}(\mu,\nu)$

$(\mathscr{P}_2(X), W_2)$ is complete and separable and

$$W_2(\mu_n,\mu) \to 0 \qquad \Leftrightarrow \qquad \begin{cases} \mu_n \rightharpoonup \mu \\ \int \mathsf{d}^2(\cdot,\bar{x})\,\mathrm{d}\mu_n \to \int \mathsf{d}^2(\cdot,\bar{x})\,\mathrm{d}\mu \end{cases}$$

The case of geodesic spaces

If (X, d) is geodesic we have

$$W_2^2(\mu,\nu) = \min \iint_0^1 |\dot\gamma_t|^2 \,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma),$$

where the min is considered among all $\boldsymbol{\pi} \in \mathscr{P}(C([0,1], X))$ such that

$$(\mathrm{e}_0)_*\boldsymbol{\pi} = \mu \qquad\qquad (\mathrm{e}_1)_*\boldsymbol{\pi} = \nu$$

A $\boldsymbol{\pi}$ realizing the minimum is be called **optimal geodesic** plan and the set of such plans is denoted by $\mathcal{GeoOpt}(\mu,\nu)$
$\boldsymbol{\pi} \in \mathcal{GeoOpt}(\mu,\nu)$ iff it is concentrated on $\mathrm{Geo}(X)$ and

$$(\mathrm{e}_0, \mathrm{e}_1)_*\boldsymbol{\pi} \in \mathcal{Opt}(\mu,\nu)$$

From measures on curves to curves of measures

Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ be with

$$\iint_0^1 |\dot{\gamma}_t|^2 \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi} < \infty$$

.

From measures on curves to curves of measures

Let $\pi \in \mathscr{P}(C([0,1],X))$ be with

$$\iint_0^1 |\dot\gamma_t|^2 \,\mathrm{d}t\,\mathrm{d}\pi < \infty$$

.

Put $\mu_t := (\mathrm{e}_t)_*\pi$ and notice that from

$$W_2^2(\mu_t,\mu_s) \leq \int \mathsf{d}^2(\gamma_t,\gamma_s)\,\mathrm{d}\pi \leq |s-t| \iint_t^s |\dot\gamma_r|^2\,\mathrm{d}r\,\mathrm{d}\pi$$

it follows that (μ_t) is W_2-AC with

$$|\dot\mu_t|^2 \leq \int |\dot\gamma_t|^2\,\mathrm{d}\pi \qquad a.e.\ t$$

The other way around

Theorem (Lisini '07) Let (μ_t) be W_2-AC with speed in $L^2(0,1)$. Then there exists $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ such that

$$(\mathrm{e}_t)_*\boldsymbol{\pi} = \mu_t \qquad \forall t \in [0,1]$$

$$\int |\dot\gamma_t|^2 \,\mathrm{d}\boldsymbol{\pi} = |\dot\mu_t|^2 \qquad a.e.\ t$$

Any such $\boldsymbol{\pi}$ will be called **lifting** of (μ_t).

The other way around

Theorem (Lisini '07) Let (μ_t) be W_2-AC with speed in $L^2(0,1)$.
Then there exists $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ such that

$$(\mathrm{e}_t)_*\boldsymbol{\pi} = \mu_t \qquad \forall t \in [0,1]$$
$$\int |\dot\gamma_t|^2 \, \mathrm{d}\boldsymbol{\pi} = |\dot\mu_t|^2 \qquad a.e.\ t$$

Any such $\boldsymbol{\pi}$ will be called **lifting** of (μ_t).

Idea of the proof: Glue together optimal plans from $\mu_{i/n}$ to $\mu_{(i+1)/n}$ and pass to the limit as $n \to \infty$

c-concave functions

$\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is c-concave provided

$$\varphi(x) = \inf_{y \in X} \frac{\mathrm{d}^2(x, y)}{2} - \psi(y)$$

for some $\psi : X \to \mathbb{R} \cup \{-\infty\}$. In this case φ is called the ***c*-transform** ψ^c of ψ.
Note that

$$\varphi^{cc} = \varphi \qquad \text{iff } \varphi \text{ is } c\text{-concave}$$

The ***c*-superdifferential** $\partial^c \varphi \subset X^2$ of φ is the set of (x, y)'s such that

$$\varphi(x) + \varphi^c(y) = \frac{\mathrm{d}^2(x, y)}{2}$$

The dual problem

Let $\mu, \nu \in \mathscr{P}_2(X)$. Maximize

$$\int \varphi \, \mathrm{d}\mu + \int \psi \, \mathrm{d}\nu$$

among all $\varphi, \psi : X \to \mathbb{R} \cup \{-\infty\}$ such that

$$\varphi(x) + \psi(y) \leq \frac{\mathrm{d}^2(x, y)}{2} \qquad \forall x, y.$$

Notice that for $\boldsymbol{\alpha} \in \mathscr{Adm}(\mu, \nu)$ we have

$$\int \varphi \, \mathrm{d}\mu + \int \psi \, \mathrm{d}\nu = \int \varphi(x) + \psi(y) \, \mathrm{d}\boldsymbol{\alpha}(x, y) \leq \frac{1}{2} \int \mathrm{d}^2(x, y) \, \mathrm{d}\boldsymbol{\alpha}(x, y)$$

Relation between primal and dual problem

Theorem (Kantorovich '48) Let $\mu, \nu \in \mathscr{P}_2(X)$. Then:

- The maximum in the dual problem is achieved by a pair of the form (φ, φ^c) for some c-concave φ
- The maximum in the dual problem is equal to the minimum of the original one (no duality gap)
- $\boldsymbol{\alpha} \in \mathcal{Adm}(\mu, \nu)$ is optimal if and only if $\mathrm{supp}(\boldsymbol{\alpha}) \subset \partial^c \varphi$ for some optimal φ

A maximizer of the dual problem is called **Kantorovich potential**

The Brenier-McCann theorem

Theorem (Brenier '87, McCann '01) Let M be a Riemannian manifold and $\mu, \nu \in \mathscr{P}_2(M)$ with $\mu \ll \mathrm{vol}$.
Then:

- There exists only one optimal plan $\boldsymbol{\alpha}$
- Such plan is induced by a map $T : M \to M$
- T is of the form $T(x) = \exp_x(-\nabla\varphi(x))$ for μ-a.e. x, where φ is any Kantorovich potential.

Moreover, there is a unique geodesic (μ_t) from μ to ν and it is given by

$$\mu_t = \exp(-t\nabla\varphi)_*\mu \qquad \forall t \in [0, 1].$$

Entropy functionals

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric space equipped with a non-negative Radon measure
We define $\mathrm{Ent}_{\mathfrak{m}} : \mathscr{P}(X) \to \mathbb{R} \cup \{+\infty\}$ as

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \displaystyle\int \rho \log \rho \, \mathrm{d}\mathfrak{m} & \text{if } \mu = \rho\mathfrak{m} \\ +\infty & \text{otherwise.} \end{cases}$$

and for $N \geq 1$ the functional $\mathrm{Ent}^N_{\mathfrak{m}} : \mathscr{P}(X) \to \mathbb{R}$ by

$$\mathrm{Ent}^N_{\mathfrak{m}}(\mu) := -N \left(\int \rho^{1-\frac{1}{N}} \, \mathrm{d}\mathfrak{m} - 1 \right) \qquad \text{for } \mu = \rho\mathfrak{m} + \mu^s$$

The curvature condition

Theorem (Sturm-VonRenesse '05) - *see also Otto-Villani and Cordero Erausquin-McCann-Schmuckenschlager*
Let M be a smooth Riemannian manifold. Then the following are equivalent:

i) The Ricci curvature of M is uniformly bounded from below by K
ii) The relative entropy functional $\mathrm{Ent}_{\mathrm{vol}}$ is K-convex on the space $(\mathscr{P}_2(M), W_2)$

The curvature condition

Theorem (Sturm-VonRenesse '05) *- see also Otto-Villani and Cordero Erausquin-McCann-Schmuckenschlager*
Let M be a smooth Riemannian manifold. Then the following are equivalent:

i) The Ricci curvature of M is uniformly bounded from below by K
ii) The relative entropy functional $\mathrm{Ent}_{\mathrm{vol}}$ is K-convex on the space $(\mathscr{P}_2(M), W_2)$

Proof: 'by direct computation'.

The curvature condition

Theorem (Sturm-VonRenesse '05) *- see also Otto-Villani and Cordero Erausquin-McCann-Schmuckenschlager*
Let M be a smooth Riemannian manifold. Then the following are equivalent:

i) The Ricci curvature of M is uniformly bounded from below by K
ii) The relative entropy functional $\mathrm{Ent}_{\mathrm{vol}}$ is K-convex on the space $(\mathscr{P}_2(M), W_2)$

Proof: 'by direct computation'.

Definition (Lott-Villani and Sturm '06) $(X, \mathsf{d}, \mathfrak{m})$ has Ricci curvature bounded from below by K if the relative entropy $\mathrm{Ent}_{\mathfrak{m}}$ is K-convex on $(\mathscr{P}_2(X), W_2)$. Called $\mathsf{CD}(K, \infty)$ spaces, in short.

The finite dimensional analogous

Theorem (Lott-Villani '05, Sturm '05)
Let M be a smooth Riemannian manifold and $N \geq 1$. Then the following are equivalent:

i) $\mathrm{Ric}_M \geq 0$ and $\dim(M) \leq N$.
ii) The Renyi entropy functional $\mathrm{Ent}_{\mathrm{vol}}^N$ is convex on the space $(\mathscr{P}_2(M), W_2)$

The finite dimensional analogous

Theorem (Lott-Villani '05, Sturm '05)
Let M be a smooth Riemannian manifold and $N \geq 1$. Then the following are equivalent:

i) $\mathrm{Ric}_M \geq 0$ and $\dim(M) \leq N$.

ii) The Renyi entropy functional $\mathrm{Ent}^N_{\mathrm{vol}}$ is convex on the space $(\mathscr{P}_2(M), W_2)$

Definition (Lott-Villani and Sturm '06) $(X, \mathsf{d}, \mathfrak{m})$ has Ricci curvature bounded from below by 0 and dimension bounded above by N if $\mathrm{Ent}^{N'}_{\mathfrak{m}}$ is convex on $(\mathscr{P}_2(X), W_2)$ for every $N' \geq N$.
Called $\mathrm{CD}(0, N)$ spaces, in short.

Note: more general $\mathrm{CD}(K, N)$ spaces can be defined along the same lines (but *not* asking for K-convexity of the Renyi entropies)

Stability of the CD condition

Theorem (Lott-Villani '05, Sturm '05)
Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ be pmGH-converging to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ and $N \in [1, \infty]$. Then:

- $\Gamma - \underline{\lim}$ inequality: for every sequence $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converging to $\mu_\infty \in \mathscr{P}(X_\infty)$ we have

$$\mathrm{Ent}^N_{\mathfrak{m}_\infty}(\mu_\infty) \leq \varliminf_{n\to\infty} \mathrm{Ent}^N_{\mathfrak{m}_n}(\mu_n).$$

- $\Gamma - \overline{\lim}$ inequality: for every $\mu_\infty \in \mathscr{P}(X_\infty)$ there is a sequence $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converging to μ_∞ such that

$$\mathrm{Ent}^N_{\mathfrak{m}_\infty}(\mu_\infty) \geq \varlimsup_{n\to\infty} \mathrm{Ent}^N_{\mathfrak{m}_n}(\mu_n).$$

Stability of the CD condition

Theorem (Lott-Villani '05, Sturm '05)
Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ be pmGH-converging to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ and $N \in [1, \infty]$. Then:

- $\Gamma - \underline{\lim}$ inequality: for every sequence $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converging to $\mu_\infty \in \mathscr{P}(X_\infty)$ we have

$$\mathrm{Ent}^N_{\mathfrak{m}_\infty}(\mu_\infty) \leq \varliminf_{n\to\infty} \mathrm{Ent}^N_{\mathfrak{m}_n}(\mu_n).$$

- $\Gamma - \overline{\lim}$ inequality: for every $\mu_\infty \in \mathscr{P}(X_\infty)$ there is a sequence $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converging to μ_∞ such that

$$\mathrm{Ent}^N_{\mathfrak{m}_\infty}(\mu_\infty) \geq \varlimsup_{n\to\infty} \mathrm{Ent}^N_{\mathfrak{m}_n}(\mu_n).$$

Corollary The $\mathrm{CD}(K, \infty)$ and $\mathrm{CD}(0, N)$ conditions are closed w.r.t. pmGH convergence.

A natural question

Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{CD}(0, N)$ space containing a (straight) line, i.e. there exists $\gamma : \mathbb{R} \to X$ such that

$$\mathsf{d}(\gamma_t, \gamma_s) = |s - t| \qquad \forall t, s \in \mathbb{R}.$$

We ask whether we deduce that X is isomorphic to a product $Y \times \mathbb{R}$ for some metric measure space Y, where the product distance is given by

$$\mathsf{d}^2_{Y\times\mathbb{R}}\big((y_1, t_1), (y_2, t_2)\big) := \mathsf{d}^2_Y(y_1, y_2) + |t_1 - t_2|^2$$

A similar question can be asked for any statement in Riemannian geometry requiring only lower bounds on the Ricci and upper bounds on the dimension.

The answer: no

Theorem (Cordero Erasquin-Sturm-Villani '08) $\mathbb{R}^d$ equipped with the Lebesgue measure and the distance coming from any norm is a $\mathrm{CD}(0, d)$ space.

PART I: Introduction

- The Cheeger-Gromoll splitting theorem
  - The original argument
  - Splitting without the Hessian
- The Riemannian curvature dimension condition
  - Metric measure spaces and their convergence
  - Optimal transport and the curvature-dimension condition
  - Functional analysis and the Riemannian curvature dimension condition

2 observations

Fact 1 A Finsler manifold is Riemannian if and only if $W^{1,2}(F)$ is a Hilbert space, in general being only Banach.
Notice that

$$\|f\|^2_{W^{1,2}(F)} = \int |f|^2 + \|\mathrm{d}f\|^2_* \, \mathrm{d}\mathfrak{m}$$

2 observations

Fact 1 A Finsler manifold is Riemannian if and only if $W^{1,2}(F)$ is a Hilbert space, in general being only Banach.
Notice that

$$\|f\|^2_{W^{1,2}(F)} = \int |f|^2 + \|\mathrm{d}f\|_*^2 \, \mathrm{d}\mathfrak{m}$$

Fact 2 Given a metric measure space $(X, \mathrm{d}, \mathfrak{m})$, it is possible to define the space $W^{1,2}(X)$, which is always a Banach space.
More on this topic later.

The proposal

Definition (G. '12) $(X, \mathsf{d}, \mathfrak{m})$ is **infinitesimally Hilbertian** provided $W^{1,2}(X)$ is Hilbert. Then define

$$\mathrm{RCD}(K,N) := \mathrm{CD}(K,N) + \text{infinitesimal Hilbertianity}$$

The case $N = \infty$ previously studied in Ambrosio, G., Savaré '11

The proposal

Definition (G. '12) $(X, \mathsf{d}, \mathfrak{m})$ is **infinitesimally Hilbertian** provided $W^{1,2}(X)$ is Hilbert. Then define

$$\mathrm{RCD}(K, N) := \mathrm{CD}(K, N) + \text{infinitesimal Hilbertianity}$$

The case $N = \infty$ previously studied in Ambrosio, G., Savaré '11

What to do in order to be sure this is a good definition:

1) Prove that the RCD condition is stable w.r.t. pmGH convergence
2) Prove some geometric consequence

Brief history of the transition from CD to RCD

G. '09 - Gradient flow of the entropy
G., Kuwada, Ohta '10 - Heat flow on Alex. spaces
Ambrosio, G., Savaré '11 - Heat flow on CD spaces
Ambrosio, G., Savaré '11- RCD(K, ∞) spaces

} Heat flow and its stability

Brief history of the transition from CD to RCD

G. '09 - Gradient flow of the entropy
G., Kuwada, Ohta '10 - Heat flow on Alex. spaces
Ambrosio, G., Savaré '11 - Heat flow on CD spaces
Ambrosio, G., Savaré '11- RCD(K, ∞) spaces

} Heat flow and its stability

G. '12 - Integration by parts and RCD(K, N) spaces
G., Mosconi '12 - Abresch-Gromoll inequality
G. '13 - Splitting on RCD$(0, N)$ spaces

} Differential calculus and geometric properties

Brief history of the transition from CD to RCD

G. '09 - Gradient flow of the entropy
G., Kuwada, Ohta '10 - Heat flow on Alex. spaces
Ambrosio, G., Savaré '11 - Heat flow on CD spaces
Ambrosio, G., Savaré '11- RCD(K, ∞) spaces

} Heat flow and its stability

G. '12 - Integration by parts and RCD(K, N) spaces
G., Mosconi '12 - Abresch-Gromoll inequality
G. '13 - Splitting on RCD$(0, N)$ spaces

} Differential calculus and geometric properties

Erbar, Kuwada, Sturm '13
Ambrosio, Mondino, Savaré '15

} Bochner inequality for $N < \infty$

Plan of the lectures

PART I: Introduction: the Cheeger-Gromoll splitting theorem and the definition of RCD spaces

PART II: Analysis on RCD spaces

PART III: Proof of the splitting in the non-smooth setting

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - 'Horizontal and vertical derivatives'
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - ‘Horizontal and vertical derivatives’
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

The basic idea

To check whether a given $f \in L^2(\mathbb{R}^d)$ belongs to $W^{1,2}(\mathbb{R}^d)$ we don't really need to know who is its (distributional) differential $\mathrm{d}f$: it is sufficient to know its modulus $|\mathrm{d}f|$ (Hajlasz).

Works by Cheeger '00 - Shanmugalingam '00 - Ambrosio, G., Savaré '11

The basic idea

To check whether a given $f \in L^2(\mathbb{R}^d)$ belongs to $W^{1,2}(\mathbb{R}^d)$ we don't really need to know who is its (distributional) differential $\mathrm{d}f$: it is sufficient to know its modulus $|\mathrm{d}f|$ (Hajlasz).

Works by Cheeger '00 - Shanmugalingam '00 - Ambrosio, G., Savaré '11

A basic example: let $f \in C^1(\mathbb{R}^d)$ and $G \in C(\mathbb{R}^d)$. Then $G \geq |\mathrm{d}f|$ iff

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t \qquad \forall \gamma \in C^1([0,1],\mathbb{R}^d).$$

Test plans

A measure $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ is called **test plan** provided

$$\iint_0^1 |\dot\gamma_t|^2 \,\mathrm{d}\boldsymbol{\pi}(\gamma) < \infty$$

$$(\mathrm{e}_t)_*\boldsymbol{\pi} \le C\mathfrak{m} \qquad \forall t \in [0,1]$$

for some $C > 0$.

Test plans

A measure $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ is called **test plan** provided

$$\iint_0^1 |\dot\gamma_t|^2 \,\mathrm{d}\boldsymbol{\pi}(\gamma) < \infty$$

$$(\mathrm{e}_t)_*\boldsymbol{\pi} \leq C\mathfrak{m} \qquad \forall t \in [0,1]$$

for some $C > 0$.

If $\boldsymbol{\pi}$ is a test plan, then so are

$$\boldsymbol{\pi}(A)^{-1}\boldsymbol{\pi}_{|_A} \qquad \forall A \subset C([0,1],X) \text{ with } \boldsymbol{\pi}(A) > 0$$

and

$$(\mathrm{restr}_t^s)_*\boldsymbol{\pi}$$

where $(\mathrm{restr}_t^s) : C([0,1],X) \to C([0,1],X)$ is given by

$$\mathrm{restr}_t^s(\gamma)_r := \gamma_{(1-r)t+rs}$$

The Sobolev class $S^2(X)$

A Borel function $f : X \to \mathbb{R}$ belongs to $S^2(X, \mathsf{d}, \mathfrak{m}) = S^2(X)$ provided there exists $G \in L^2(X)$, $G \geq 0$ so that

$$\int |f(\gamma_1) - f(\gamma_0)| \, \mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 G(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma) \tag{3}$$

for every test plan $\boldsymbol{\pi}$. Such G is called **weak upper gradient**.

The Sobolev class $S^2(X)$

A Borel function $f : X \to \mathbb{R}$ belongs to $S^2(X, \mathsf{d}, \mathfrak{m}) = S^2(X)$ provided there exists $G \in L^2(X)$, $G \geq 0$ so that

$$\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 G(\gamma_t)|\dot\gamma_t| \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}(\boldsymbol{\gamma}) \qquad (3)$$

for every test plan $\boldsymbol{\pi}$. Such G is called **weak upper gradient**.

(3) holds iff for $\boldsymbol{\pi}$-a.e. γ the function $t \mapsto f(\gamma_t)$ is in $W^{1,1}(0,1)$ and

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\Big|(t) \leq G(\gamma_t)|\dot\gamma_t| \qquad a.e.\ t.$$

The minimal weak upper gradient

Let $f \in S^2(X)$.

From the integral characterization we see that the set of w.u.g.'s is closed in $L^2(X)$

From the differential characterization we see that $\min\{G_1, G_2\}$ is a w.u.g. if G_1, G_2 are.

Thus there exists a w.u.g. which is $\mathfrak{m}$-a.e. smaller than any other. We shall call it **minimal weak upper gradient** and denote it by $|\mathrm{d}f|$.

Calculus rules for $|\mathrm{d}f|$

Lower semicontinuity:

$$\left.\begin{array}{rl} (f_n) & \subset S^2(X) \\ f_n & \to f \qquad \mathfrak{m}-a.e. \\ |\mathrm{d}f_n| & \rightharpoonup G \text{ in } L^2(\mathfrak{m}) \end{array}\right\} \qquad \Rightarrow \qquad \left\{\begin{array}{rl} f & \in S^2(X) \\ |\mathrm{d}f| & \leq G \end{array}\right.$$

Subadditivity: $|\mathrm{d}(\alpha f + \beta g)| \leq |\alpha||\mathrm{d}f| + |\beta||\mathrm{d}g| \qquad \mathfrak{m}-a.e.$

Locality: $|\mathrm{d}f| = |\mathrm{d}g| \qquad \mathfrak{m}-a.e.\ on\ \{f = g\}$

Chain rule: $|\mathrm{d}(\varphi \circ f)| = |\varphi'| \circ f\,|\mathrm{d}f|,$ for $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz

Leibniz rule: $|\mathrm{d}(fg)| \leq |f||\mathrm{d}g| + |g||\mathrm{d}f|,$ for $f, g \in S^2 \cap L^\infty(X)$

The Sobolev space $W^{1,2}$

We put $W^{1,2}(X) := L^2 \cap S^2(X)$ and endow it with the norm

$$\|f\|^2_{W^{1,2}} := \|f\|^2_{L^2} + \||\mathrm{d}f|\|^2_{L^2}$$

This is always a Banach space: completeness comes from the lower semicontinuity of $|\mathrm{d}f|$

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - 'Horizontal and vertical derivatives'
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

The setting

From now on we restrict the attention to infinitesimally Hilbertian spaces, i.e. so that $W^{1,2}(X)$ is Hilbert

The object $\langle \nabla f, \nabla g \rangle$

For $f, g \in S^2(X)$ we put

$$\langle \nabla f, \nabla g \rangle := \inf_{\varepsilon > 0} \frac{|\mathrm{d}(g + \varepsilon f)|^2 - |\mathrm{d}g|^2}{2\varepsilon}$$

The object $\langle \nabla f, \nabla g \rangle$

For $f, g \in S^2(X)$ we put

$$\langle \nabla f, \nabla g \rangle := \inf_{\varepsilon > 0} \frac{|\mathrm{d}(g + \varepsilon f)|^2 - |\mathrm{d}g|^2}{2\varepsilon}$$

Note:

- this definition makes sense even on Finsler setting, in this case the right hand side gives

$$\mathrm{d}f(\nabla g)$$

- Coincides with the object $\Gamma(f, g)$ known in the context of Dirichlet forms

Basic properties of $\langle \nabla f, \nabla g \rangle$

Theorem (Ambrosio, G.,Savaré '11- G. '12)
Let $(X, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian. Then

$$\text{the } \inf_{\varepsilon>0} \text{ in the definition of } \langle \nabla f, \nabla g \rangle \text{ can be replaced by } \lim_{\varepsilon\to 0},$$

the limit being both in $L^1(X)$ and $\mathfrak{m}$-a.e..
Moreover, $(f, g) \mapsto \langle \nabla f, \nabla g \rangle$ is bilinear, symmetric and satisfies:

- 'Locality'

$$\langle \nabla f, \nabla g \rangle = \langle \nabla f', \nabla g' \rangle \qquad \mathfrak{m} - a.e. \text{ on } \{f = f'\} \cap \{g = g'\}$$

- 'Cauchy-Schwarz'

$$|\langle \nabla f, \nabla g \rangle| \leq |\mathrm{d}f||\mathrm{d}g| \qquad \langle \nabla f, \nabla f \rangle = |\mathrm{d}f|^2$$

- 'Chain rule'

$$\langle \nabla(\varphi \circ f), \nabla g \rangle = \varphi' \circ f \langle \nabla f, \nabla g \rangle$$

- 'Leibniz rule'

$$\langle \nabla(f_1 f_2), \nabla g \rangle = f_1 \langle \nabla f_2, \nabla g \rangle + f_2 \langle \nabla f_1, \nabla g \rangle$$

Proof of chain rule

$$\langle \nabla(\varphi \circ f), \nabla g \rangle = \varphi' \circ f \langle \nabla f, \nabla g \rangle$$

For $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz

Proof of chain rule

$$\langle \nabla(\varphi \circ f), \nabla g \rangle = \varphi' \circ f \langle \nabla f, \nabla g \rangle$$

For $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz

- By linearity, obvious if φ is linear

Proof of chain rule

$$\langle \nabla(\varphi \circ f), \nabla g \rangle = \varphi' \circ f \langle \nabla f, \nabla g \rangle$$

For $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz

- By linearity, obvious if φ is linear
- Since $|\mathrm{d}(f + C)| = |\mathrm{d}f|$, obvious if φ is affine

Proof of chain rule

$$\langle \nabla(\varphi \circ f), \nabla g \rangle = \varphi' \circ f \langle \nabla f, \nabla g \rangle$$

For $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz

- By linearity, obvious if φ is linear
- Since $|\mathrm{d}(f + C)| = |\mathrm{d}f|$, obvious if φ is affine
- By locality, obvious if φ is piecewise affine

Proof of chain rule

$$\langle \nabla(\varphi \circ f), \nabla g \rangle = \varphi' \circ f \langle \nabla f, \nabla g \rangle$$

For $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz

- By linearity, obvious if φ is linear
- Since $|\mathrm{d}(f + C)| = |\mathrm{d}f|$, obvious if φ is affine
- By locality, obvious if φ is piecewise affine
- By approximation, conclude for general φ Lipschitz

Proof of Leibniz rule

$$\langle \nabla(f_1 f_2), \nabla g \rangle = f_1 \langle \nabla f_2, \nabla g \rangle + f_2 \langle \nabla f_1, \nabla g \rangle$$

for $f_1, f_2, g \in W^{1,2}(X)$ and $f_1, f_2 \in L^\infty(X)$.

Proof of Leibniz rule

$$\langle \nabla(f_1 f_2), \nabla g \rangle = f_1 \langle \nabla f_2, \nabla g \rangle + f_2 \langle \nabla f_1, \nabla g \rangle$$

for $f_1, f_2, g \in W^{1,2}(X)$ and $f_1, f_2 \in L^\infty(X)$.

It is safe to assume $f_1, f_2 \geq 1$.

Proof of Leibniz rule

$$\langle \nabla(f_1 f_2), \nabla g \rangle = f_1 \langle \nabla f_2, \nabla g \rangle + f_2 \langle \nabla f_1, \nabla g \rangle$$

for $f_1, f_2, g \in W^{1,2}(X)$ and $f_1, f_2 \in L^\infty(X)$.

It is safe to assume $f_1, f_2 \geq 1$. Then

$$\begin{aligned} \frac{1}{f_1 f_2} \langle \nabla(f_1 f_2), \nabla g \rangle &= \langle \nabla(\log(f_1 f_2)), \nabla g \rangle \\ &= \langle \nabla(\log(f_1) + \log(f_2)), \nabla g \rangle \\ &= \langle \nabla(\log(f_1)), \nabla g \rangle + \langle \nabla(\log(f_2)), \nabla g \rangle \\ &= \frac{1}{f_1} \langle \nabla f_1, \nabla g \rangle + \frac{1}{f_2} \langle \nabla f_2, \nabla g \rangle \end{aligned}$$

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - 'Horizontal and vertical derivatives'
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

Plan representing gradients

For any $f \in S^2(X)$ and $\boldsymbol{\pi}$ test plan we have

$$\varlimsup_{t\downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \,\mathrm{d}\boldsymbol{\pi} \leq \frac{1}{2} \int |\mathrm{d}f|^2(\gamma_0) \,\mathrm{d}\boldsymbol{\pi} + \varlimsup_{t\downarrow 0} \frac{1}{2t} \iint_0^t |\dot{\gamma}_s|^2 \,\mathrm{d}s \,\mathrm{d}\boldsymbol{\pi}$$

Plan representing gradients

For any $f \in S^2(X)$ and $\boldsymbol{\pi}$ test plan we have

$$\varlimsup_{t\downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi} \leq \frac{1}{2} \int |\mathrm{d}f|^2(\gamma_0) \, \mathrm{d}\boldsymbol{\pi} + \varlimsup_{t\downarrow 0} \frac{1}{2t} \iint_0^t |\dot{\gamma}_s|^2 \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}$$

Definition (G. '12) We say that $\boldsymbol{\pi}$ represents ∇f provided

$$\varliminf_{t\downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi} \geq \frac{1}{2} \int |\mathrm{d}f|^2(\gamma_0) \, \mathrm{d}\boldsymbol{\pi} + \varlimsup_{t\downarrow 0} \frac{1}{2t} \iint_0^t |\dot{\gamma}_s|^2 \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}$$

Note: in the smooth case this would mean $\gamma_0' = \nabla f(\gamma_0)$ for $\boldsymbol{\pi}$-a.e. γ

Existence results

Theorem (Ambrosio, G., Savaré '11) Let:

- (μ_t) be a W_2-geodesic such that $\mu_t \leq C\mathfrak{m}$,
- $\boldsymbol{\pi}$ a lifting of it
- $\varphi \in S^2(X)$ a Kantorovich potential associated to μ_0, μ_1.

Then $\boldsymbol{\pi}$ represents $\nabla(-\varphi)$.

Existence results

Theorem (Ambrosio, G., Savaré '11) Let:

- (μ_t) be a W_2-geodesic such that $\mu_t \leq C\mathfrak{m}$,
- $\boldsymbol{\pi}$ a lifting of it
- $\varphi \in S^2(X)$ a Kantorovich potential associated to μ_0, μ_1.

Then $\boldsymbol{\pi}$ represents $\nabla(-\varphi)$.

Proof: 'Follows the definitions'

Existence results

Theorem (Ambrosio, G., Savaré '11) Let:

- (μ_t) be a W_2-geodesic such that $\mu_t \leq C\mathfrak{m}$,
- $\boldsymbol{\pi}$ a lifting of it
- $\varphi \in S^2(X)$ a Kantorovich potential associated to μ_0, μ_1.

Then $\boldsymbol{\pi}$ represents $\nabla(-\varphi)$.

Proof: 'Follows the definitions'

Note: In the Riemannian case McCann's theorem grants that for $\boldsymbol{\pi}$-a.e. γ we have $\gamma_t = \exp(-t\nabla\varphi(\gamma_0))$

Existence results

Theorem (Ambrosio, G., Savaré '11) Let:

- (μ_t) be a W_2-geodesic such that $\mu_t \leq C\mathfrak{m}$,
- $\boldsymbol{\pi}$ a lifting of it
- $\varphi \in S^2(X)$ a Kantorovich potential associated to μ_0, μ_1.

Then $\boldsymbol{\pi}$ represents $\nabla(-\varphi)$.

Proof: 'Follows the definitions'

Note: In the Riemannian case McCann's theorem grants that for $\boldsymbol{\pi}$-a.e. γ we have $\gamma_t = \exp(-t\nabla\varphi(\gamma_0))$

Theorem (G. '12) Let $f \in S^2(X)$ and $\mu \in \mathscr{P}(X)$ with $\mu \leq C\mathfrak{m}$ for some $C > 0$. Then there exists $\boldsymbol{\pi}$ representing ∇f such that

$$(\mathrm{e}_0)_*\boldsymbol{\pi} = \mu$$

Horizontal and vertical derivatives

Theorem (Ambrosio,G.,Savaré '11 - G. '12) Let $f, g \in S^2(X)$ and $\boldsymbol{\pi}$ representing ∇g.
Then

$$\lim_{t \downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi} =$$

Horizontal and vertical derivatives

Theorem (Ambrosio,G.,Savaré '11 - G. '12) Let $f, g \in S^2(X)$ and $\boldsymbol{\pi}$ representing ∇g.
Then

$$\lim_{t \downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi} = \int \langle \nabla f, \nabla g \rangle (\gamma_0) \, \mathrm{d}\boldsymbol{\pi}$$

Horizontal and vertical derivatives

Theorem (Ambrosio,G.,Savaré '11 - G. '12) Let $f, g \in S^2(X)$ and $\boldsymbol{\pi}$ representing ∇g.
Then

$$\lim_{t\downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi} = \int \langle \nabla f, \nabla g \rangle (\gamma_0) \, \mathrm{d}\boldsymbol{\pi}$$

Proof: For $\varepsilon \in \mathbb{R}$ we have

$$\overline{\lim_{t\downarrow 0}} \int \frac{(g+\varepsilon f)(\gamma_t) - (g+\varepsilon f)(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi} \leq \frac{1}{2} \int |\mathrm{d}(g+\varepsilon f)|^2(\gamma_0) \, \mathrm{d}\boldsymbol{\pi} + \overline{\lim_{t\downarrow 0}} \frac{1}{2t} \iint_0^t |\dot{\gamma}_s|^2 \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}$$

and

$$\underline{\lim_{t\downarrow 0}} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi} \geq \frac{1}{2} \int |\mathrm{d}g|^2(\gamma_0) \, \mathrm{d}\boldsymbol{\pi} + \overline{\lim_{t\downarrow 0}} \frac{1}{2t} \iint_0^t |\dot{\gamma}_s|^2 \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}$$

Subtracting the second from the first we obtain

$$\overline{\lim_{t\downarrow 0}} \, \varepsilon \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi} \leq \frac{1}{2} \int |\mathrm{d}(g+\varepsilon f)|^2(\gamma_0) - |\mathrm{d}g|^2(\gamma_0) \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}\boldsymbol{\pi}$$

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - 'Horizontal and vertical derivatives'
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

The product $X \times \mathbb{R}$

Let $(X, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian and consider the product space $X \times \mathbb{R}$ with the distance

$$\mathsf{d}_{X\times\mathbb{R}}\big((x_1, t_1), (x_2, t_2)\big)^2 := \mathsf{d}_X^2(x_1, x_2) + |t_1 - t_2|^2$$

and the measure $\mathfrak{m}_X \times \mathcal{L}^1$.

The product $X \times \mathbb{R}$

Let $(X, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian and consider the product space $X \times \mathbb{R}$ with the distance

$$\mathsf{d}_{X\times\mathbb{R}}\big((x_1,t_1),(x_2,t_2)\big)^2 := \mathsf{d}_X^2(x_1,x_2) + |t_1 - t_2|^2$$

and the measure $\mathfrak{m}_X \times \mathcal{L}^1$.

For $f : X \times \mathbb{R} \to \mathbb{R}$ define

$$f^{(x)}(t) := f(t,x) \qquad \forall x \in X$$
$$f^{(t)}(x) := f(t,x) \qquad \forall t \in \mathbb{R}$$

Sobolev functions on $X \times \mathbb{R}$

Theorem (G., Han '12 *see also Ambrosio, G., Savaré '11*) TFAE:

i) $f \in S^2(X \times \mathbb{R})$

ii) we have

ii-a) for $\mathfrak{m}$-a.e. x it holds $f^{(x)} \in S^2(\mathbb{R})$ and

$$\iint |\mathrm{d}f^{(x)}|^2_{\mathbb{R}}(t)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}t < \infty$$

ii-b) for $\mathcal{L}^1$-a.e. t it holds $f^{(t)} \in S^2(X)$ and

$$\iint |\mathrm{d}f^{(t)}|^2_{X}(x)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}t < \infty$$

In this case it holds

$$|\mathrm{d}f|^2_{X\times\mathbb{R}}(x,t) = |\mathrm{d}f^{(x)}|^2_{\mathbb{R}}(t) + |\mathrm{d}f^{(t)}|^2_X(x) \qquad \mathfrak{m} \times \mathcal{L}^1 - a.e.\ (t,x).$$

Moreover, linear combinations of functions of the form $g(x)h(t)$ are dense in $W^{1,2}(X' \times \mathbb{R})$

Sobolev functions on $X \times \mathbb{R}$

Theorem (G., Han '12 *see also Ambrosio, G., Savaré '11*) TFAE:

i) $f \in S^2(X \times \mathbb{R})$

ii) we have

ii-a) for $\mathfrak{m}$-a.e. x it holds $f^{(x)} \in S^2(\mathbb{R})$ and

$$\iint |\mathrm{d}f^{(x)}|^2_{\mathbb{R}}(t)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}t < \infty$$

ii-b) for $\mathcal{L}^1$-a.e. t it holds $f^{(t)} \in S^2(X)$ and

$$\iint |\mathrm{d}f^{(t)}|^2_{X}(x)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}t < \infty$$

In this case it holds

$$|\mathrm{d}f|^2_{X\times\mathbb{R}}(x,t) = |\mathrm{d}f^{(x)}|^2_{\mathbb{R}}(t) + |\mathrm{d}f^{(t)}|^2_{X}(x) \qquad \mathfrak{m}\times\mathcal{L}^1 - a.e.\ (t,x).$$

Moreover, linear combinations of functions of the form $g(x)h(t)$ are dense in $W^{1,2}(X' \times \mathbb{R})$

Note: the analogous result with $\mathbb{R}$ replaced by a generic Y is unknown.

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - 'Horizontal and vertical derivatives'
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

The Laplacian

The space $D(\Delta) \subset W^{1,2}(X)$ is the set of f's for which there exists a function $h \in L^2(X)$ such that

$$\int gh \, \mathrm{d}\mathfrak{m} = -\int \langle \nabla g, \nabla f \rangle \, \mathrm{d}\mathfrak{m} \qquad \forall g \in W^{1,2}(X)$$

Such h is unique and denoted as Δf

The Laplacian

The space $D(\Delta) \subset W^{1,2}(X)$ is the set of f's for which there exists a function $h \in L^2(X)$ such that

$$\int gh\,\mathrm{d}\mathfrak{m} = -\int \langle \nabla g, \nabla f \rangle\,\mathrm{d}\mathfrak{m} \qquad \forall g \in W^{1,2}(X)$$

Such h is unique and denoted as Δf

Such Laplacian coincides with:

- The infinitesimal generator of the Dirichlet form

$$E(f) := \frac{1}{2}\int |\mathrm{d}f|^2\,\mathrm{d}\mathfrak{m}$$

- The only element of the subdifferential $\partial^- E(f)$.

The heat flow in $L^2(X)$

Theorem For every $f \in L^2(X)$ there exists a unique $t \mapsto \mathsf{h}_t(f) \in L^2(X)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{h}_t f = \Delta \mathsf{h}_t f$$

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - 'Horizontal and vertical derivatives'
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

Characterizing gradient flows of K-convex functions on $\mathbb{R}^d$

Let $E : \mathbb{R}^d \to \mathbb{R}$ be convex. Then $v \in \partial^- E(x)$ iff

$$\langle v, y - x \rangle \leq E(y) - E(x) - \frac{K}{2}|y - x|^2 \qquad \forall y$$

Characterizing gradient flows of K-convex functions on $\mathbb{R}^d$

Let $E : \mathbb{R}^d \to \mathbb{R}$ be convex. Then $v \in \partial^- E(x)$ iff

$$\langle v, y - x \rangle \leq E(y) - E(x) - \frac{K}{2}|y - x|^2 \qquad \forall y$$

Since we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|x_t - y|^2 = \langle x_t', x_t - y \rangle$$

we see that $x_t' \in -\partial^- E(x_t)$ for a.e. t iff

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|x_t - y|^2 \leq E(y) - E(x) - \frac{K}{2}|y - x|^2 \qquad \forall y \;\; a.e.\; t > 0$$

The Evolution Variational Inequality

Definition (Ambrosio, G., Savaré '04) Let (Y, d_Y) be a metric space and $E : Y \to \mathbb{R} \cup \{+\infty\}$ be l.s.c.. We say that $t \mapsto x_t \in Y$ is an EVI_K gradient flow of E provided it is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathrm{d}_Y^2(x_t, y) + E(x_t) + \frac{K}{2}\mathrm{d}^2(x_t, y) \leq E(y) \qquad \forall y \in Y \;\; a.e.\; t > 0$$

The Evolution Variational Inequality

Definition (Ambrosio, G., Savaré '04) Let (Y, d_Y) be a metric space and $E : Y \to \mathbb{R} \cup \{+\infty\}$ be l.s.c.. We say that $t \mapsto x_t \in Y$ is an EVI_K gradient flow of E provided it is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathrm{d}^2_Y(x_t, y) + E(x_t) + \frac{K}{2}\mathrm{d}^2(x_t, y) \leq E(y) \qquad \forall y \in Y \;\; a.e.\; t > 0$$

- Extensively studied by Savaré and coauthors
- Having EVI_K-grad.fl. encodes both K-convexity of the functional and some Riemannian-like property of the space
- We shall consider gradient flows of the Boltzmann entropy in this sense.

EVI_K implies contractivity

Let $(x_t), (y_t)$ be EVI_K-grad.fl of E. Then

$$\mathsf{d}_Y(x_t, y_t) \leq e^{-Kt}\mathsf{d}_Y(x_0, y_0)$$

EVI_K implies contractivity

Let $(x_t), (y_t)$ be EVI_K-grad.fl of E. Then

$$\mathsf{d}_Y(x_t, y_t) \le e^{-Kt}\mathsf{d}_Y(x_0, y_0)$$

Indeed, up to technicalities:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathsf{d}_Y^2(x_t, y_t)_{|t=t_0} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathsf{d}_Y^2(x_t, y_{t_0})_{|t=t_0} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathsf{d}_Y^2(x_{t_0}, y_t)_{|t=t_0} \\
&\le \Big(E(y_{t_0}) - E(x_{t_0}) - \frac{K}{2}\mathsf{d}_Y^2(x_{t_0}, y_{t_0})\Big) \\
&\qquad + \Big(E(x_{t_0}) - E(y_{t_0}) - \frac{K}{2}\mathsf{d}_Y^2(x_{t_0}, y_{t_0})\Big) \\
&= -K\,\mathsf{d}_Y^2(x_{t_0}, y_{t_0})
\end{aligned}$$

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - 'Horizontal and vertical derivatives'
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

$RCD(K, \infty)$ spaces

Theorem (Ambrosio, G., Savaré '12 see also Ambrosio, G., Mondino, Rajala '13)
TFAE:

i) $(X, \mathsf{d}, \mathfrak{m})$ is an infinitesimally Hilbertian $CD(K, \infty)$ space

ii) Any measure $\mu \in \mathscr{P}_2(X)$ is the starting point of an EVI_K-grad.fl. of $\mathrm{Ent}_{\mathfrak{m}}$ in $(\mathscr{P}_2(X), W_2)$

If these hold and $\rho \in L^2(X)$ is such that $\mu = \rho\mathfrak{m} \in \mathscr{P}_2(X)$, then the EVI_K-grad.fl. of $\mathrm{Ent}_{\mathfrak{m}}$ is given by

$$\mu_t := (\mathsf{h}_t \rho)\mathfrak{m}$$

'Proof' of $(i) \Rightarrow (ii)$

(1/3)

Let $\rho \in L^2(X)$ be such that $\mu = \rho\mathfrak{m} \in \mathscr{P}_2(X)$, put $\mu_t := (\mathsf{h}_t\rho)\mathfrak{m}$ and pick $\nu \in \mathscr{P}_2(X)$.

‘Proof’ of $(i) \Rightarrow (ii)$

Let $\rho \in L^2(X)$ be such that $\mu = \rho\mathfrak{m} \in \mathscr{P}_2(X)$, put $\mu_t := (\mathsf{h}_t\rho)\mathfrak{m}$ and pick $\nu \in \mathscr{P}_2(X)$.

Fix t_0 and let φ_{t_0} be a Kantorovich potential for (μ_t, ν).

‘Proof’ of $(i) \Rightarrow (ii)$

(1/3)

Let $\rho \in L^2(X)$ be such that $\mu = \rho\mathfrak{m} \in \mathscr{P}_2(X)$, put $\mu_t := (\mathsf{h}_t\rho)\mathfrak{m}$ and pick $\nu \in \mathscr{P}_2(X)$.

Fix t_0 and let φ_{t_0} be a Kantorovich potential for (μ_t, ν).

Then

$$\frac{1}{2}W_2^2(\mu_{t_0}, \nu) = \int \varphi_{t_0}\mathsf{h}_{t_0}\rho\,\mathrm{d}\mathfrak{m} + \int \varphi_{t_0}^c\,\mathrm{d}\nu$$

$$\frac{1}{2}W_2^2(\mu_t, \nu) \geq \int \varphi_{t_0}\mathsf{h}_t\rho\,\mathrm{d}\mathfrak{m} + \int \varphi_{t_0}^c\,\mathrm{d}\nu \qquad \forall t > 0.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu)_{|t=t_0} = \frac{\mathrm{d}}{\mathrm{d}t}\int \varphi_{t_0}\mathsf{h}_t\rho\,\mathrm{d}\mathfrak{m}_{|t=t_0} = \int \varphi_{t_0}\Delta\mathsf{h}_{t_0}\rho\,\mathrm{d}\mathfrak{m}$$

‘Proof’ of $(i) \Rightarrow (ii)$

(2/3)

Let $s \mapsto \nu_s = \eta_s \mathfrak{m}$ be a W_2-geodesic connecting μ_{t_0} to ν.

‘Proof’ of $(i) \Rightarrow (ii)$

(2/3)

Let $s \mapsto \nu_s = \eta_s \mathfrak{m}$ be a W_2-geodesic connecting μ_{t_0} to ν. Let’s pretend that $\mathrm{Ent}_{\mathfrak{m}}$ is K-convex along (ν_s) and that $\nu_s \leq C\mathfrak{m}$ for some $C >$ for every $s \in [0, 1]$.

‘Proof’ of $(i) \Rightarrow (ii)$

Let $s \mapsto \nu_s = \eta_s \mathfrak{m}$ be a W_2-geodesic connecting μ_{t_0} to ν. Let's pretend that $\mathrm{Ent}_{\mathfrak{m}}$ is K-convex along (ν_s) and that $\nu_s \leq C\mathfrak{m}$ for some $C >$ for every $s \in [0, 1]$.
Let $\boldsymbol{\pi}$ be a lifting of (ν_s). Then

$$\begin{aligned}\mathrm{Ent}_{\mathfrak{m}}(\nu_s) - \mathrm{Ent}_{\mathfrak{m}}(\nu_0) &\geq \int \log(\mathsf{h}_{t_0}\rho)(\eta_s - \rho_{t_0})\,\mathrm{d}\mathfrak{m} \\ &= \int \log(\mathsf{h}_{t_0}\rho)(\gamma_s) - \log(\mathsf{h}_{t_0}\rho)(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\end{aligned}$$

'Proof' of $(i) \Rightarrow (ii)$

(2/3)

Let $s \mapsto \nu_s = \eta_s \mathfrak{m}$ be a W_2-geodesic connecting μ_{t_0} to ν. Let's pretend that $\mathrm{Ent}_{\mathfrak{m}}$ is K-convex along (ν_s) and that $\nu_s \leq C\mathfrak{m}$ for some $C >$ for every $s \in [0,1]$.

Let $\boldsymbol{\pi}$ be a lifting of (ν_s). Then

$$\begin{aligned}
\mathrm{Ent}_{\mathfrak{m}}(\nu_s) - \mathrm{Ent}_{\mathfrak{m}}(\nu_0) &\geq \int \log(\mathrm{h}_{t_0}\rho)(\eta_s - \rho_{t_0})\,\mathrm{d}\mathfrak{m} \\
&= \int \log(\mathrm{h}_{t_0}\rho)(\gamma_s) - \log(\mathrm{h}_{t_0}\rho)(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma)
\end{aligned}$$

Recalling that $\boldsymbol{\pi}$ represents $\nabla(-\varphi_{t_0})$ we get

$$\begin{aligned}
\varliminf_{s\downarrow 0} \frac{\mathrm{Ent}_{\mathfrak{m}}(\nu_s) - \mathrm{Ent}_{\mathfrak{m}}(\nu_0)}{s} &\geq -\int \langle \nabla \log(\mathrm{h}_{t_0}\rho), \nabla\varphi_{t_0}\rangle(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= -\int \langle \nabla \log(\mathrm{h}_{t_0}\rho), \nabla\varphi_{t_0}\rangle \mathrm{h}_{t_0}\rho\,\mathrm{d}\mathfrak{m} \\
&= -\int \langle \nabla \mathrm{h}_{t_0}\rho, \nabla\varphi_{t_0}\rangle\,\mathrm{d}\mathfrak{m}
\end{aligned}$$

‘Proof’ of $(i) \Rightarrow (ii)$

(3/3)

We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu)\big|_{t=t_0} &= \int \varphi_{t_0}\Delta \mathsf{h}_{t_0}\rho\,\mathrm{d}\mathfrak{m} \\
&= -\int \langle \nabla \log(\mathsf{h}_{t_0}\rho), \nabla\varphi_{t_0}\rangle \mathsf{h}_{t_0}\rho\,\mathrm{d}\mathfrak{m} \\
&\leq \varliminf_{s\downarrow 0} \frac{\mathrm{Ent}_{\mathfrak{m}}(\nu_s)-\mathrm{Ent}_{\mathfrak{m}}(\nu_0)}{s} \\
&\leq \mathrm{Ent}_{\mathfrak{m}}(\nu)-\mathrm{Ent}_{\mathfrak{m}}(\mu_{t_0})-\frac{K}{2}W_2^2(\mu_{t_0},\nu)
\end{aligned}$$

Stability of the $\mathrm{RCD}(K,\infty)$ condition

Pass to the limit in the integral formulation of the EVI_K:

$$\frac{W_2^2(\mu_s,\nu)-W_2^2(\mu_t,\nu)}{2}+\int_t^s \mathrm{Ent}_{\mathfrak{m}}(\mu_r)+\frac{K}{2}W_2^2(\mu_r,\nu)\,\mathrm{d}r \le (s-t)\mathrm{Ent}_{\mathfrak{m}}(\nu)$$

for any $0\le t\le s$ and $\nu\in\mathscr{P}_2(X)$.

Stability of the RCD(K, ∞) condition

Pass to the limit in the integral formulation of the EVI_K:

$$\frac{W_2^2(\mu_s,\nu) - W_2^2(\mu_t,\nu)}{2} + \int_t^s \mathrm{Ent}_{\mathfrak{m}}(\mu_r) + \frac{K}{2} W_2^2(\mu_r,\nu)\,\mathrm{d}r \leq (s-t)\mathrm{Ent}_{\mathfrak{m}}(\nu)$$

for any $0 \leq t \leq s$ and $\nu \in \mathscr{P}_2(X)$.

The stability of the

$$\mathrm{RCD}(K,N) = \mathrm{RCD}(K,\infty) \cap \mathrm{CD}(K,N)$$

condition also follows

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - 'Horizontal and vertical derivatives'
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

The Bochner inequality in the nonsmooth setting

Theorem (Ambrosio, G., Savaré '11 and G., Kuwada, Ohta '10)
Let $(X, \mathsf{d}, \mathfrak{m})$ be a RCD(K, ∞) space. Then the inequality

$$\Delta \frac{|\mathrm{d}f|^2}{2} \geq \langle \nabla f, \nabla \Delta f \rangle + K|\mathrm{d}f|^2$$

holds in the weak sense, i.e.

$$\frac{1}{2} \int \Delta g |\mathrm{d}f|^2 \, \mathrm{d}\mathfrak{m} \geq \int g \big(\nabla f, \nabla \Delta f \rangle + K|\mathrm{d}f|^2 \big) \, \mathrm{d}\mathfrak{m}$$

for every $f \in D(\Delta)$ with $\Delta f \in W^{1,2}(X)$ and $g \in L^\infty(X) \cap D(\Delta)$ with $\Delta g \in L^\infty(X)$.

The Bochner inequality in the nonsmooth setting

Theorem (Ambrosio, G., Savaré '11 and G., Kuwada, Ohta '10)
Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{RCD}(K, \infty)$ space. Then the inequality

$$\Delta \frac{|\mathrm{d}f|^2}{2} \geq \langle \nabla f, \nabla \Delta f \rangle + K|\mathrm{d}f|^2$$

holds in the weak sense, i.e.

$$\frac{1}{2} \int \Delta g |\mathrm{d}f|^2 \, \mathrm{d}\mathfrak{m} \geq \int g \big(\nabla f, \nabla \Delta f \rangle + K|\mathrm{d}f|^2 \big) \, \mathrm{d}\mathfrak{m}$$

for every $f \in D(\Delta)$ with $\Delta f \in W^{1,2}(X)$ and $g \in L^\infty(X) \cap D(\Delta)$ with $\Delta g \in L^\infty(X)$.

Note: the class of f, g's in the statement is dense in $W^{1,2}(X)$ (and thus also in $L^2(X)$)

How to obtain the Bochner inequality on RCD(K, ∞) spaces

RCD(K, ∞) condition

How to obtain the Bochner inequality on RCD(K, ∞) spaces

RCD(K, ∞) condition

$\Downarrow$ via the EVI_K

$$W_2(\mathsf{h}_t(\mu), \mathsf{h}_t(\nu)) \le e^{-Kt} W_2(\mu, \nu)$$

How to obtain the Bochner inequality on RCD(K,∞) spaces

$$\mathrm{RCD}(K,\infty) \text{ condition}$$

$$\Downarrow \qquad \text{via the } EVI_K$$

$$W_2(\mathsf{h}_t(\mu),\mathsf{h}_t(\nu)) \le e^{-Kt} W_2(\mu,\nu)$$

$$\Downarrow \qquad \text{via Kuwada's duality}$$

$$|\mathrm{d}(\mathsf{h}_t f)|^2 \le e^{-2Kt}\, \mathsf{h}_t(|\mathrm{d}f|^2)$$

How to obtain the Bochner inequality on $\mathrm{RCD}(K,\infty)$ spaces

$$\mathrm{RCD}(K,\infty) \;\; \text{condition}$$

$$\Downarrow \qquad \text{via the } EVI_K$$

$$W_2(\mathsf{h}_t(\mu),\mathsf{h}_t(\nu)) \le e^{-Kt} W_2(\mu,\nu)$$

$$\Downarrow \qquad \text{via Kuwada's duality}$$

$$|\mathrm{d}(\mathsf{h}_t f)|^2 \le e^{-2Kt}\,\mathsf{h}_t(|\mathrm{d}f|^2)$$

$$\Downarrow \qquad \text{differentiating at } t=0$$

$$\Delta\frac{|\mathrm{d}f|^2}{2} \ge \langle \nabla f, \nabla\Delta f\rangle + |\mathrm{d}f|^2$$

How to obtain the Bochner inequality on $\mathrm{RCD}(K,\infty)$ spaces

$$
\begin{aligned}
\mathrm{RCD}(K,\infty) &\quad \text{condition} \\
&\Downarrow \qquad \text{via the } EVI_K \\
W_2(\mathsf{h}_t(\mu),\mathsf{h}_t(\nu)) &\le e^{-Kt} W_2(\mu,\nu) \\
&\Downarrow \qquad \text{via Kuwada's duality} \\
|\mathrm{d}(\mathsf{h}_t f)|^2 &\le e^{-2Kt}\,\mathsf{h}_t(|\mathrm{d}f|^2) \\
&\Downarrow \qquad \text{differentiating at } t=0 \\
\Delta\frac{|\mathrm{d}f|^2}{2} &\ge \langle \nabla f, \nabla\Delta f\rangle + |\mathrm{d}f|^2
\end{aligned}
$$

Note:

- The reverse implications also hold (Ambrosio, G., Savaré 12)
- This scheme also gives Bochner inequality for $N < \infty$ (Erbar, Kuwada, Sturm '13 and Ambrosio, Mondino, Savaré '15).

Rough idea for Kuwada's duality

Notice that

$$\mathsf{h}_t f(x) = \int f \, \mathrm{d}\mathsf{h}_t(\delta_x)$$

so that

$$|\mathsf{h}_t f(y) - \mathsf{h}_t f(x)| = \left| \int f \, \mathrm{d}\mathsf{h}_t(\delta_y) - \int f \, \mathrm{d}\mathsf{h}_t(\delta_x) \right|$$

Rough idea for Kuwada's duality

Notice that

$$\mathsf{h}_t f(x) = \int f \,\mathrm{d}\mathsf{h}_t(\delta_x)$$

so that

$$\begin{aligned} |\mathsf{h}_t f(y) - \mathsf{h}_t f(x)| &= \Big| \int f \,\mathrm{d}\mathsf{h}_t(\delta_y) - \int f \,\mathrm{d}\mathsf{h}_t(\delta_x) \Big| \\ &= \Big| \int f(w) - f(z) \,\mathrm{d}\boldsymbol{\alpha}_t(w,z) \Big| \qquad \boldsymbol{\alpha}_t \in \mathcal{O}pt(\mathsf{h}_t(\delta_x), \mathsf{h}_t(\delta_y)) \end{aligned}$$

Rough idea for Kuwada's duality

Notice that

$$\mathsf{h}_t f(x) = \int f \, \mathrm{d}\mathsf{h}_t(\delta_x)$$

so that

$$\begin{aligned} |\mathsf{h}_t f(y) - \mathsf{h}_t f(x)| &= \left| \int f \, \mathrm{d}\mathsf{h}_t(\delta_y) - \int f \, \mathrm{d}\mathsf{h}_t(\delta_x) \right| \\ &= \left| \int f(w) - f(z) \, \mathrm{d}\boldsymbol{\alpha}_t(w, z) \right| \qquad \boldsymbol{\alpha}_t \in \mathcal{O}pt(\mathsf{h}_t(\delta_x), \mathsf{h}_t(\delta_y)) \\ &\leq \int \frac{|f(w) - f(z)|}{\mathsf{d}(z, w)} \mathsf{d}(z, w) \, \mathrm{d}\boldsymbol{\alpha}_t(w, z) \end{aligned}$$

Rough idea for Kuwada's duality

Notice that

$$\mathsf{h}_t f(x) = \int f \, \mathrm{d}\mathsf{h}_t(\delta_x)$$

so that

$$\begin{aligned}
|\mathsf{h}_t f(y) - \mathsf{h}_t f(x)| &= \left| \int f \, \mathrm{d}\mathsf{h}_t(\delta_y) - \int f \, \mathrm{d}\mathsf{h}_t(\delta_x) \right| \\
&= \left| \int f(w) - f(z) \, \mathrm{d}\boldsymbol{\alpha}_t(w,z) \right| \qquad \boldsymbol{\alpha}_t \in \mathcal{O}pt(\mathsf{h}_t(\delta_x), \mathsf{h}_t(\delta_y)) \\
&\leq \int \frac{|f(w) - f(z)|}{\mathsf{d}(z,w)} \mathsf{d}(z,w) \, \mathrm{d}\boldsymbol{\alpha}_t(w,z) \\
&\leq \sqrt{\int \frac{|f(w) - f(z)|^2}{\mathsf{d}^2(z,w)} \, \mathrm{d}\boldsymbol{\alpha}_t(w,z)} \, W_2(\mathsf{h}_t(\delta_x), \mathsf{h}_t(\delta_y))
\end{aligned}$$

Rough idea for Kuwada's duality

Notice that

$$\mathsf{h}_t f(x) = \int f \, \mathrm{d}\mathsf{h}_t(\delta_x)$$

so that

$$\begin{aligned}
|\mathsf{h}_t f(y) - \mathsf{h}_t f(x)| &= \left| \int f \, \mathrm{d}\mathsf{h}_t(\delta_y) - \int f \, \mathrm{d}\mathsf{h}_t(\delta_x) \right| \\
&= \left| \int f(w) - f(z) \, \mathrm{d}\boldsymbol{\alpha}_t(w, z) \right| \qquad \boldsymbol{\alpha}_t \in \mathcal{O}pt(\mathsf{h}_t(\delta_x), \mathsf{h}_t(\delta_y)) \\
&\leq \int \frac{|f(w) - f(z)|}{\mathsf{d}(z, w)} \mathsf{d}(z, w) \, \mathrm{d}\boldsymbol{\alpha}_t(w, z) \\
&\leq \sqrt{\int \frac{|f(w) - f(z)|^2}{\mathsf{d}^2(z, w)} \, \mathrm{d}\boldsymbol{\alpha}_t(w, z)} \, W_2(\mathsf{h}_t(\delta_x), \mathsf{h}_t(\delta_y)) \\
&\leq e^{-Kt} \mathsf{d}(x, y) \sqrt{\int \frac{|f(w) - f(z)|^2}{\mathsf{d}^2(z, w)} \, \mathrm{d}\boldsymbol{\alpha}_t(w, z)}
\end{aligned}$$

Rough idea for Kuwada's duality

Notice that

$$\mathsf{h}_t f(x) = \int f \, \mathrm{d}\mathsf{h}_t(\delta_x)$$

so that

$$\begin{aligned}
|\mathsf{h}_t f(y) - \mathsf{h}_t f(x)| &= \left| \int f \, \mathrm{d}\mathsf{h}_t(\delta_y) - \int f \, \mathrm{d}\mathsf{h}_t(\delta_x) \right| \\
&= \left| \int f(w) - f(z) \, \mathrm{d}\boldsymbol{\alpha}_t(w,z) \right| \qquad \boldsymbol{\alpha}_t \in \mathcal{O}pt(\mathsf{h}_t(\delta_x), \mathsf{h}_t(\delta_y)) \\
&\leq \int \frac{|f(w) - f(z)|}{\mathsf{d}(z,w)} \mathsf{d}(z,w) \, \mathrm{d}\boldsymbol{\alpha}_t(w,z) \\
&\leq \sqrt{\int \frac{|f(w) - f(z)|^2}{\mathsf{d}^2(z,w)} \, \mathrm{d}\boldsymbol{\alpha}_t(w,z)} W_2(\mathsf{h}_t(\delta_x), \mathsf{h}_t(\delta_y)) \\
&\leq e^{-Kt} \mathsf{d}(x,y) \sqrt{\int \frac{|f(w) - f(z)|^2}{\mathsf{d}^2(z,w)} \, \mathrm{d}\boldsymbol{\alpha}_t(w,z)} \\
&\sim e^{-Kt} \mathsf{d}(x,y) \sqrt{\int \mathrm{lip}^2(f) \, \mathrm{d}\mathsf{h}_t(\delta_x)} \\
&= e^{-Kt} \mathsf{d}(x,y) \sqrt{\mathsf{h}_t(\mathrm{lip}^2(f))(x)}
\end{aligned}$$

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - ‘Horizontal and vertical derivatives’
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

Optimal maps

Theorem (G., Rajala, Sturm '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be $\mathrm{RCD}(K, \infty)$ and $\mu_0 = \rho_0 \mathfrak{m}, \mu_1 = \rho_1 \mathfrak{m} \in \mathscr{P}_2(X)$.

Optimal maps

Theorem (G., Rajala, Sturm '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be $\mathsf{RCD}(K, \infty)$ and $\mu_0 = \rho_0 \mathfrak{m}, \mu_1 = \rho_1 \mathfrak{m} \in \mathscr{P}_2(X)$.
Then:

- There exists only one optimal plan between them
- such plan is induced by a map T
- for μ_0-a.e x the geodesic connecting x to $T(x)$ is unique.

Optimal maps

Theorem (G., Rajala, Sturm '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be RCD(K, ∞) and $\mu_0 = \rho_0 \mathfrak{m}, \mu_1 = \rho_1 \mathfrak{m} \in \mathscr{P}_2(X)$.
Then:

- There exists only one optimal plan between them
- such plan is induced by a map T
- for μ_0-a.e x the geodesic connecting x to $T(x)$ is unique.

In particular, there exists only one $\boldsymbol{\pi} \in \mathcal{G}eo\mathcal{O}pt(\mu_0, \mu_1)$ and for $\boldsymbol{\pi}$-a.e. γ the map

$$t \quad \mapsto \quad \log(\rho_t)(\gamma_t)$$

is K-convex.

Optimal maps

Theorem (G., Rajala, Sturm '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be RCD(K, ∞) and $\mu_0 = \rho_0 \mathfrak{m}, \mu_1 = \rho_1 \mathfrak{m} \in \mathscr{P}_2(X)$.
Then:

- There exists only one optimal plan between them
- such plan is induced by a map T
- for μ_0-a.e x the geodesic connecting x to $T(x)$ is unique.

In particular, there exists only one $\boldsymbol{\pi} \in \mathcal{G}eo\mathcal{O}pt(\mu_0, \mu_1)$ and for $\boldsymbol{\pi}$-a.e. γ the map

$$t \quad \mapsto \quad \log(\rho_t)(\gamma_t)$$

is K-convex.
It follows that if ρ_0, ρ_1 are in L^∞ and with bounded support, then $\boldsymbol{\pi}$ is a test plan
In the finite dimensional case, the thesis holds for general μ_1.

Optimal maps

Theorem (G., Rajala, Sturm '13) Let $(X, \mathrm{d}, \mathfrak{m})$ be $\mathrm{RCD}(K, \infty)$ and $\mu_0 = \rho_0 \mathfrak{m}, \mu_1 = \rho_1 \mathfrak{m} \in \mathscr{P}_2(X)$.
Then:

- There exists only one optimal plan between them
- such plan is induced by a map T
- for μ_0-a.e x the geodesic connecting x to $T(x)$ is unique.

In particular, there exists only one $\boldsymbol{\pi} \in \mathcal{GeoOpt}(\mu_0, \mu_1)$ and for $\boldsymbol{\pi}$-a.e. γ the map

$$t \quad \mapsto \quad \log(\rho_t)(\gamma_t)$$

is K-convex.
It follows that if ρ_0, ρ_1 are in L^∞ and with bounded support, then $\boldsymbol{\pi}$ is a test plan
In the finite dimensional case, the thesis holds for general μ_1.

This can be read as a non-branching result
The proof has to do with stronger geodesic convexity of the entropy on $\mathrm{RCD}(K, \infty)$ spaces.

The Sobolev-to-Lipschitz property

Theorem (Ambrosio, G., Savaré '11) Let $(X, \mathrm{d}, \mathfrak{m})$ be $\mathrm{RCD}(K, \infty)$ and $f \in W^{1,2}(X)$ be such that $|\mathrm{d}f| \in L^\infty(X)$.
Then f has a Lipschitz representative $\tilde{f}$ such that

$$\mathrm{Lip}(\tilde{f}) \leq \||\mathrm{d}f|\|_{L^\infty}$$

The Sobolev-to-Lipschitz property

Theorem (Ambrosio, G., Savaré '11) Let $(X, \mathsf{d}, \mathfrak{m})$ be $\mathsf{RCD}(K, \infty)$ and $f \in W^{1,2}(X)$ be such that $|\mathrm{d}f| \in L^\infty(X)$.
Then f has a Lipschitz representative $\tilde{f}$ such that

$$\mathrm{Lip}(\tilde{f}) \leq \||\mathrm{d}f|\|_{L^\infty}$$

Proof Let $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ be with bounded support and density, and $\boldsymbol{\pi}$ the only element of $\mathcal{GeoOpt}(\mu_0, \mu_1)$ a lifting of it.
Then $\boldsymbol{\pi}$ is a test plan and thus

$$\Big|\int f\,\mathrm{d}\mu_1 - \int f\,\mathrm{d}\mu_0\Big| = \Big|\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}\Big| \leq \iint_0^1 |\mathrm{d}f|(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}$$

$$\leq \||\mathrm{d}f|\|_{L^\infty}\sqrt{\iint_0^1 |\dot{\gamma}_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}} = \||\mathrm{d}f|\|_{L^\infty} W_2(\mu_0, \mu_1)$$

Isomorphism by duality with Sobolev functions

Theorem (G. '12) Let $(X_1, \mathsf{d}_1, \mathfrak{m}_1),\ (X_2, \mathsf{d}_2, \mathfrak{m}_2)$ be two $\mathrm{RCD}(K, \infty)$ spaces and $T : X_1 \to X_2$ Borel and a.e. invertible.
Then T is, up to a modification in a negligible set, a measure preserving isometry iff

$$\|f \circ T\|_{W^{1,2}(X_1)} = \|f\|_{W^{1,2}(X_2)} \qquad \forall f : X_2 \to \mathbb{R} \text{ Borel}$$

Isomorphism by duality with Sobolev functions

Theorem (G. '12) Let $(X_1, \mathsf{d}_1, \mathfrak{m}_1)$, $(X_2, \mathsf{d}_2, \mathfrak{m}_2)$ be two RCD(K, ∞) spaces and $T : X_1 \to X_2$ Borel and a.e. invertible.
Then T is, up to a modification in a negligible set, a measure preserving isometry iff

$$\|f \circ T\|_{W^{1,2}(X_1)} = \|f\|_{W^{1,2}(X_2)} \qquad \forall f : X_2 \to \mathbb{R} \text{ Borel}$$

Compare with: Let (X_1, d_1), (X_2, d_2) be two metric spaces and $T : X_1 \to X_2$ invertible. Then T is an isometry iff

$$\mathrm{Lip}_{X_1}(f \circ T) = \mathrm{Lip}_{X_2}(f) \qquad \forall f : X_2 \to \mathbb{R}.$$

Proof assuming that the reference measures are finite (1/3)

The identity

$$\|1+f\|^2_{W^{1,2}} - \|f\|^2_{W^{1,2}} = 2\int f \,\mathrm{d}\mathfrak{m}$$

shows that T is measure preserving and thus that

$$\int |\mathrm{d}(f\circ T)|^2 \,\mathrm{d}\mathfrak{m}_1 = \int |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{m}_2$$

which by polarization gives

$$\int \langle \nabla\tilde{f}, \nabla\tilde{g}\rangle \,\mathrm{d}\mathfrak{m}_1 = \int \langle \nabla f, \nabla g\rangle \,\mathrm{d}\mathfrak{m}_2$$

for $\tilde{f} := f\circ T$, $\tilde{g} := g\circ T$.

Proof assuming that the reference measures are finite (2/3)

Therefore we have

$$\int \tilde{g}|\mathrm{d}\tilde{f}|^2 \, \mathrm{d}\mathfrak{m}_1 = \int \langle \nabla(\tilde{g}\tilde{f}), \nabla \tilde{f} \rangle - \tfrac{1}{2}\langle \nabla \tilde{g}, \nabla(\tilde{f}^2) \rangle \, \mathrm{d}\mathfrak{m}_1$$
$$= \int \langle \nabla(gf), \nabla f \rangle - \tfrac{1}{2}\langle \nabla g, \nabla(f^2) \rangle \, \mathrm{d}\mathfrak{m}_2$$
$$= \int g|\mathrm{d}f|^2 \, \mathrm{d}\mathfrak{m}_2 = \int \tilde{g}|\mathrm{d}f|^2 \circ T \, \mathrm{d}\mathfrak{m}_1$$

Which leads to

$$|\mathrm{d}(f \circ T)| = |\mathrm{d}f| \circ T \qquad \mathfrak{m}_1 - a.e..$$

Proof assuming that the reference measures are finite (3/3)

Let $(x_n) \subset X_2$ be countable and dense, put $f_n := \mathrm{d}_2(\cdot, x_n)$ and notice that from

$$|\mathrm{d}(f_n \circ T)| = |\mathrm{d}f_n| \circ T \leq \mathrm{Lip}(f_n) = 1 \qquad \mathfrak{m} - a.e.$$

and the Sobolev-to-Lipschitz property we deduce that $f_n \circ T$ has a 1-Lipschitz representative.
Thus there is $N \subset X_1$ with $\mathfrak{m}_1(N) = 0$ so that the restriction of $f_n \circ T$ to $X_1 \setminus N$ is 1-Lipschitz for every $n \in \mathbb{N}$. Hence

$$\mathrm{d}_2(T(x), T(y)) = \sup_n |f_n(T(x)) - f_n(T(y))| \leq \mathrm{d}_1(x, y)$$

That is, T has a 1-Lipschitz representative. Exchanging the roles of X_1 and X_2 we conclude

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - 'Horizontal and vertical derivatives'
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

Definition of measure-valued Laplacian

Definition (G. '12) Let $(X, \mathsf{d}, \mathfrak{m})$ be proper and infinitesimally Hilbertian, $\Omega \subset X$ open and $f \in W^{1,2}(\Omega)$.
We say that $f \in D(\underline{\Delta}, \Omega)$ provided there is a locally finite Borel measure μ on Ω such that

$$\int g \, \mathrm{d}\mu = -\int \langle \nabla g, \nabla f \rangle \, \mathrm{d}\mathfrak{m} \qquad \forall g \in \mathrm{LIP}_c(\Omega)$$

In this case μ is clearly unique and will be denoted by $\underline{\Delta}_{|\Omega} f$
IF $\Omega = X$ we write $\underline{\Delta} f$

Definition of measure-valued Laplacian

Definition (G. '12) Let $(X, \mathsf{d}, \mathfrak{m})$ be proper and infinitesimally Hilbertian, $\Omega \subset X$ open and $f \in W^{1,2}(\Omega)$.
We say that $f \in D(\underline{\Delta}, \Omega)$ provided there is a locally finite Borel measure μ on Ω such that

$$\int g \, \mathrm{d}\mu = -\int \langle \nabla g, \nabla f \rangle \, \mathrm{d}\mathfrak{m} \qquad \forall g \in \mathrm{LIP}_c(\Omega)$$

In this case μ is clearly unique and will be denoted by $\underline{\Delta}_{|_\Omega} f$
IF $\Omega = X$ we write $\underline{\Delta} f$

Note:

- $\underline{\Delta} f = h\mathfrak{m}$ for some $h \in L^2(X)$ iff $f \in D(\Delta)$ and in this case $h = \Delta f$
- The natural chain and Leibniz rules are in place for $\underline{\Delta}$, as a consequence of those for $\langle \nabla f, \nabla g \rangle$

Measure-valued Laplacian and energy minimization

Theorem (G. '12 - G., Mondino '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be proper and infinitesimally Hilbertian, $\Omega \subset X$ open and $f \in W^{1,2}(\Omega)$.
Then TFAE:

- $f \in D(\underline{\Delta}, \Omega)$ and $\underline{\Delta}_{|\Omega} f \geq 0$
- $$\int_\Omega |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{m} \leq \int_\Omega |\mathrm{d}g|^2 \,\mathrm{d}\mathfrak{m}$$

  for every $g \leq f$ such that $\mathrm{supp}(f - g) \subset\subset \Omega$

Measure-valued Laplacian and energy minimization

Theorem (G. '12 - G., Mondino '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be proper and infinitesimally Hilbertian, $\Omega \subset X$ open and $f \in W^{1,2}(\Omega)$.
Then TFAE:

- $f \in D(\underline{\Delta}, \Omega)$ and $\underline{\Delta}_{|\Omega} f \geq 0$
- $$\int_\Omega |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{m} \leq \int_\Omega |\mathrm{d}g|^2 \,\mathrm{d}\mathfrak{m}$$
  for every $g \leq f$ such that $\mathrm{supp}(f-g) \subset\subset \Omega$

Proof: Same as in $\mathbb{R}^d$: write the Euler equation for the minimization problem. Uses some form of density of Lipschitz functions in $W^{1,2}(X)$

Laplacian comparison estimates

Theorem (G. '12) Let $(X, \mathsf{d}, \mathfrak{m})$ be RCD$(0, N)$ and $x \in X$.
Then $\frac{1}{2}\mathsf{d}^2(\cdot, x) \in D(\underline{\Delta})$ and

$$\underline{\Delta}\frac{1}{2}\mathsf{d}^2(\cdot, x) \leq N\mathfrak{m}$$

Laplacian comparison estimates

Theorem (G. '12) Let $(X, \mathsf{d}, \mathfrak{m})$ be RCD$(0, N)$ and $x \in X$. Then $\frac{1}{2}\mathsf{d}^2(\cdot, x) \in D(\underline{\Delta})$ and

$$\underline{\Delta}\frac{1}{2}\mathsf{d}^2(\cdot, x) \leq N\mathfrak{m}$$

Main idea of the proof: For any $\mu \in \mathscr{P}_2(X)$, the function $\frac{\mathsf{d}^2(\cdot,x)}{2}$ is a Kantorovich potential for μ, δ_x

Analogous sharp inequality true for general curvature bounds

Idea of the proof of Laplacian comparison

Pick $f \in \mathrm{LIP}_c(X)$ non-negative, put $\mu = \rho_0 \mathfrak{m} = c f^{\frac{N}{N-1}} \mathfrak{m} \in \mathscr{P}_2(X)$.

Idea of the proof of Laplacian comparison

Pick $f \in \mathrm{LIP}_c(X)$ non-negative, put $\mu = \rho_0 \mathfrak{m} = c f^{\frac{N}{N-1}} \mathfrak{m} \in \mathscr{P}_2(X)$.
Let $t \mapsto \mu_t = \rho_t \mathfrak{m}$ the W_2-geodesic from μ to δ_x and $\boldsymbol{\pi}$ its lifting.

Idea of the proof of Laplacian comparison

Pick $f \in \mathrm{LIP}_c(X)$ non-negative, put $\mu = \rho_0 \mathfrak{m} = c f^{\frac{N}{N-1}} \mathfrak{m} \in \mathscr{P}_2(X)$.
Let $t \mapsto \mu_t = \rho_t \mathfrak{m}$ the W_2-geodesic from μ to δ_x and $\boldsymbol{\pi}$ its lifting.
Then for $u(z) = -z^{1-\frac{1}{N}}$ we have

$$c^{1-\frac{1}{N}} \int f \,\mathrm{d}\mathfrak{m} = -\mathrm{Ent}^N_{\mathfrak{m}}(\mu) \geq \frac{\mathrm{Ent}^N_{\mathfrak{m}}(\mu_t) - \mathrm{Ent}^N_{\mathfrak{m}}(\mu)}{t}$$

Idea of the proof of Laplacian comparison

Pick $f \in \mathrm{LIP}_c(X)$ non-negative, put $\mu = \rho_0 \mathfrak{m} = c f^{\frac{N}{N-1}} \mathfrak{m} \in \mathscr{P}_2(X)$.
Let $t \mapsto \mu_t = \rho_t \mathfrak{m}$ the W_2-geodesic from μ to δ_x and $\boldsymbol{\pi}$ its lifting.
Then for $u(z) = -z^{1-\frac{1}{N}}$ we have

$$\begin{aligned} c^{1-\frac{1}{N}} \int f \, \mathrm{d}\mathfrak{m} &= -\mathrm{Ent}_{\mathfrak{m}}^{N}(\mu) \geq \frac{\mathrm{Ent}_{\mathfrak{m}}^{N}(\mu_t) - \mathrm{Ent}_{\mathfrak{m}}^{N}(\mu)}{t} \\ &\geq \frac{1}{t} \int u'(\rho)(\rho_t - \rho) \, \mathrm{d}\mathfrak{m} \end{aligned}$$

Idea of the proof of Laplacian comparison

Pick $f \in \mathrm{LIP}_c(X)$ non-negative, put $\mu = \rho_0 \mathfrak{m} = c f^{\frac{N}{N-1}} \mathfrak{m} \in \mathscr{P}_2(X)$.
Let $t \mapsto \mu_t = \rho_t \mathfrak{m}$ the W_2-geodesic from μ to δ_x and $\boldsymbol{\pi}$ its lifting.
Then for $u(z) = -z^{1-\frac{1}{N}}$ we have

$$\begin{aligned} c^{1-\frac{1}{N}} \int f \, \mathrm{d}\mathfrak{m} &= -\mathrm{Ent}^N_{\mathfrak{m}}(\mu) \geq \frac{\mathrm{Ent}^N_{\mathfrak{m}}(\mu_t) - \mathrm{Ent}^N_{\mathfrak{m}}(\mu)}{t} \\ &\geq \frac{1}{t} \int u'(\rho)(\rho_t - \rho) \, \mathrm{d}\mathfrak{m} = \frac{1}{t} \int u'(\rho)(\gamma_t) - u'(\rho)(\gamma_0) \, \mathrm{d}\boldsymbol{\pi} \end{aligned}$$

Since $\boldsymbol{\pi}$ represents $\nabla(-\frac{\mathrm{d}^2(\cdot,x)}{2})$ we get

$$\lim_{t\downarrow 0} \int u'(\rho)(\gamma_t) - u'(\rho)(\gamma_0) \, \mathrm{d}\boldsymbol{\pi} = -\int \langle \nabla u'(\rho_0), \nabla \frac{\mathrm{d}^2(\cdot,x)}{2} \rangle (\gamma_0) \, \mathrm{d}\boldsymbol{\pi}$$

Idea of the proof of Laplacian comparison

Pick $f \in \mathrm{LIP}_c(X)$ non-negative, put $\mu = \rho_0 \mathfrak{m} = c f^{\frac{N}{N-1}} \mathfrak{m} \in \mathscr{P}_2(X)$.
Let $t \mapsto \mu_t = \rho_t \mathfrak{m}$ the W_2-geodesic from μ to δ_x and $\boldsymbol{\pi}$ its lifting.
Then for $u(z) = -z^{1-\frac{1}{N}}$ we have

$$\begin{aligned} c^{1-\frac{1}{N}} \int f \, \mathrm{d}\mathfrak{m} &= -\mathrm{Ent}^N_{\mathfrak{m}}(\mu) \geq \frac{\mathrm{Ent}^N_{\mathfrak{m}}(\mu_t) - \mathrm{Ent}^N_{\mathfrak{m}}(\mu)}{t} \\ &\geq \frac{1}{t} \int u'(\rho)(\rho_t - \rho) \, \mathrm{d}\mathfrak{m} = \frac{1}{t} \int u'(\rho)(\gamma_t) - u'(\rho)(\gamma_0) \, \mathrm{d}\boldsymbol{\pi} \end{aligned}$$

Since $\boldsymbol{\pi}$ represents $\nabla(-\frac{\mathrm{d}^2(\cdot,x)}{2})$ we get

$$\begin{aligned} \lim_{t\downarrow 0} \int u'(\rho)(\gamma_t) - u'(\rho)(\gamma_0) \, \mathrm{d}\boldsymbol{\pi} &= -\int \langle \nabla u'(\rho_0), \nabla \tfrac{\mathrm{d}^2(\cdot,x)}{2} \rangle (\gamma_0) \, \mathrm{d}\boldsymbol{\pi} \\ &= -\int \langle \nabla u'(\rho), \nabla \tfrac{\mathrm{d}^2(\cdot,x)}{2} \rangle \rho \, \mathrm{d}\mathfrak{m} \end{aligned}$$

Idea of the proof of Laplacian comparison

Pick $f \in \mathrm{LIP}_c(X)$ non-negative, put $\mu = \rho_0 \mathfrak{m} = c f^{\frac{N}{N-1}} \mathfrak{m} \in \mathscr{P}_2(X)$.
Let $t \mapsto \mu_t = \rho_t \mathfrak{m}$ the W_2-geodesic from μ to δ_x and $\boldsymbol{\pi}$ its lifting.
Then for $u(z) = -z^{1-\frac{1}{N}}$ we have

$$
\begin{aligned}
c^{1-\frac{1}{N}} \int f \,\mathrm{d}\mathfrak{m} &= -\mathrm{Ent}^N_{\mathfrak{m}}(\mu) \geq \frac{\mathrm{Ent}^N_{\mathfrak{m}}(\mu_t) - \mathrm{Ent}^N_{\mathfrak{m}}(\mu)}{t} \\
&\geq \frac{1}{t} \int u'(\rho)(\rho_t - \rho) \,\mathrm{d}\mathfrak{m} = \frac{1}{t} \int u'(\rho)(\gamma_t) - u'(\rho)(\gamma_0) \,\mathrm{d}\boldsymbol{\pi}
\end{aligned}
$$

Since $\boldsymbol{\pi}$ represents $\nabla(-\frac{\mathrm{d}^2(\cdot,x)}{2})$ we get

$$
\begin{aligned}
\lim_{t\downarrow 0} \int u'(\rho)(\gamma_t) - u'(\rho)(\gamma_0) \,\mathrm{d}\boldsymbol{\pi} &= -\int \langle \nabla u'(\rho_0), \nabla \tfrac{\mathrm{d}^2(\cdot,x)}{2} \rangle (\gamma_0) \,\mathrm{d}\boldsymbol{\pi} \\
&= -\int \langle \nabla u'(\rho), \nabla \tfrac{\mathrm{d}^2(\cdot,x)}{2} \rangle \rho \,\mathrm{d}\mathfrak{m} \\
&= -\frac{c^{1-\frac{1}{N}}}{N} \int \langle \nabla f, \nabla \tfrac{\mathrm{d}^2(\cdot,x)}{2} \rangle \,\mathrm{d}\mathfrak{m}
\end{aligned}
$$

PART II: analysis on RCD spaces

- Sobolev calculus on general metric measure spaces
  - The space $W^{1,2}(X)$ and the quantity $|\mathrm{d}f|$
  - Infinitesimally Hilbertian spaces and the quantity $\langle \nabla f, \nabla g \rangle$
  - ‘Horizontal and vertical derivatives’
  - Sobolev calculus on product spaces
- Heat flow as gradient flow
  - The L^2 viewpoint
  - The W_2 viewpoint
  - Identification and stability
- Bochner inequality
- Some qualitative consequences of the lower Ricci bounds
- Laplacian comparison
- Maximum principle

The weak maximum principle

Theorem Let $(X, \mathsf{d}, \mathfrak{m})$ be $\mathrm{RCD}(K, \infty)$, $\Omega \subset X$ open and $f \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$.
Assume that $f \in D(\underline{\Delta}, \Omega)$ with $\underline{\Delta}_{|\Omega} f \geq 0$.
Then

$$\sup_{\Omega} f \leq \sup_{\partial\Omega} f$$

The weak maximum principle

Theorem Let $(X, \mathsf{d}, \mathfrak{m})$ be $\mathrm{RCD}(K, \infty)$, $\Omega \subset X$ open and $f \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$.
Assume that $f \in D(\underline{\Delta}, \Omega)$ with $\underline{\Delta}_{|\Omega} f \geq 0$.
Then

$$\sup_{\Omega} f \leq \sup_{\partial\Omega} f$$

Proof: look at the picture and use, e.g., the Sobolev-to-Lipschitz property

The strong maximum principle

Theorem Let $(X, \mathsf{d}, \mathfrak{m})$ be $\mathrm{RCD}(K, N)$, $\Omega \subset X$ open connected and $f \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$.
Assume that $f \in D(\underline{\Delta}, \Omega)$ with $\underline{\Delta}_{|\Omega} f \geq 0$ and that

$$f(x_0) = \sup_{\partial\Omega} f \qquad \text{for some } x_0 \in \Omega.$$

Then f is constant in Ω

The strong maximum principle

Theorem Let $(X, \mathsf{d}, \mathfrak{m})$ be $\mathrm{RCD}(K, N)$, $\Omega \subset X$ open connected and $f \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$.
Assume that $f \in D(\underline{\Delta}, \Omega)$ with $\underline{\Delta}_{|_\Omega} f \geq 0$ and that

$$f(x_0) = \sup_{\partial\Omega} f \qquad \text{for some } x_0 \in \Omega.$$

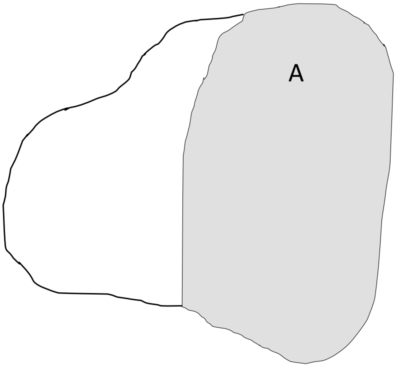
Then f is constant in Ω

Proof: by contradiction using a perturbation argument and the weak maximum principle.
Let

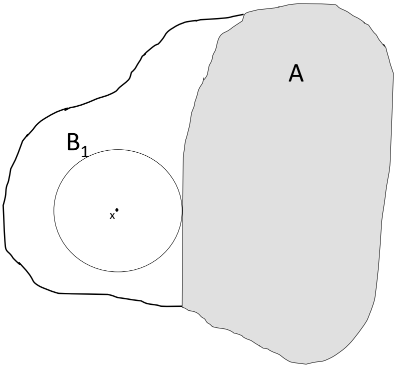
$$A := \left\{ x \in \Omega \;:\; f(x) = \sup_{\partial\Omega} f \right\}$$

and assume that $A \neq \emptyset$ and $A \neq \Omega$

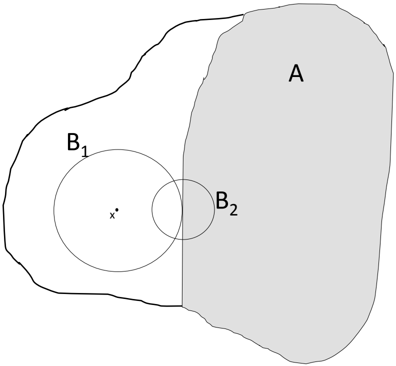
Proof of the strong maximum principle



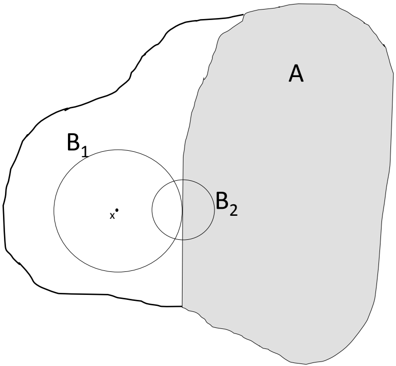
Proof of the strong maximum principle



Proof of the strong maximum principle

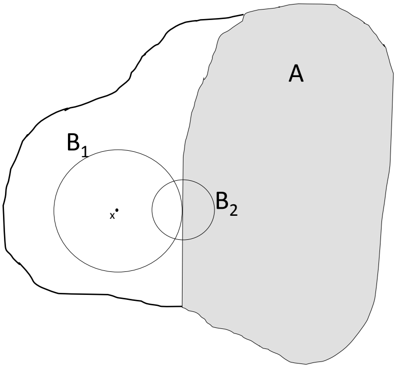


Proof of the strong maximum principle



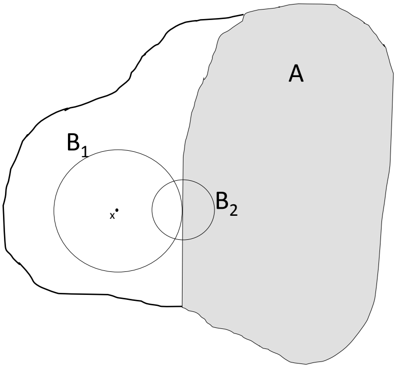
Find $u \in C(\overline{\Omega}) \cap D(\underline{\Delta}, \Omega)$ such that $\{u \geq 0\} = \overline{B_1}$ and $\underline{\Delta}|_{B_2} u \geq 0$.

Proof of the strong maximum principle



Find $u \in C(\overline{\Omega}) \cap D(\underline{\Delta}, \Omega)$ such that $\{u \geq 0\} = \overline{B_1}$ and $\underline{\Delta}|_{B_2} u \geq 0$.
The contradiction comes looking at $f + \varepsilon u$ for $0 < \varepsilon \ll 1$.

Proof of the strong maximum principle



Find $u \in C(\overline{\Omega}) \cap D(\underline{\Delta}, \Omega)$ such that $\{u \geq 0\} = \overline{B_1}$ and $\underline{\Delta}|_{B_2} u \geq 0$.
The contradiction comes looking at $f + \varepsilon u$ for $0 < \varepsilon \ll 1$.
The function $u := \exp(-C\mathrm{d}^2(\cdot, x)) - C'$ for $1 \ll C$ and appropriate C' does the job

Plan of the lectures

PART I: Introduction: the Cheeger-Gromoll splitting theorem and the definition of RCD spaces

PART II: Analysis on RCD spaces

PART III: Proof of the splitting in the non-smooth setting

Statement of the result

Theorem (G. '13) Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD(0, N)$ space containing a (straight) line, i.e. a map $\gamma : \mathbb{R} \to X$ such that

$$\mathsf{d}(\gamma_s, \gamma_t) = |s - t| \qquad \forall t, s \in \mathbb{R}.$$

Then there is a space $(X', \mathsf{d}', \mathfrak{m}')$ such that

$$(X, \mathsf{d}, \mathfrak{m}) \quad \text{is isomorphic to} \quad (X' \times \mathbb{R}, \mathsf{d}' \otimes \mathsf{d}_{\rm Eucl}, \mathfrak{m}' \times \mathcal{L}^1)$$

where

$$(\mathsf{d}' \otimes \mathsf{d}_{\rm Eucl})\big((x', t), (y', s)\big) := \sqrt{\mathsf{d}'(x', y')^2 + |t - s|^2}$$

Moreover:

- If $N \geq 2$ then $(X', \mathsf{d}', \mathfrak{m}')$ is an $RCD(0, N - 1)$ space
- If $N \in [1, 2)$ then X' contains only one point

PART III: Proof of the splitting in the non-smooth setting

- Busemann's function and harmonicity
- 'Gradient flow' of b and measure preservation
- 'Gradient flow' of b and isometries
- The quotient space and its isometric embedding
- Splitting
- Dimension reduction

PART III: Proof of the splitting in the non-smooth setting

- Busemann's function and harmonicity
- 'Gradient flow' of b and measure preservation
- 'Gradient flow' of b and isometries
- The quotient space and its isometric embedding
- Splitting
- Dimension reduction

The Busemann functions

As in the smooth case we define:

$$\mathrm{b}^{+}(x) := \lim_{t\to+\infty} t - \mathsf{d}(x, \gamma_t)$$

$$\mathrm{b}^{-}(x) := \lim_{t\to+\infty} t - \mathsf{d}(x, \gamma_{-t})$$

Laplacian comparison estimate and its use

As in the smooth case we start from

$$\underline{\Delta}\frac{\mathsf{d}^2(\cdot,\bar{x})}{2} \leq N\mathfrak{m} \qquad \text{on } X$$

to deduce that

$$\underline{\Delta}|_{X\setminus\{\gamma_t\}}\mathsf{d}(\cdot,\gamma_t) \leq \frac{N}{\mathsf{d}(\cdot,\gamma_t)}\mathfrak{m}$$

Laplacian comparison estimate and its use

As in the smooth case we start from

$$\underline{\Delta}\frac{\mathsf{d}^2(\cdot,\bar{x})}{2} \le N\mathfrak{m} \qquad \textit{on } X$$

to deduce that

$$\underline{\Delta}|_{X\setminus\{\gamma_t\}}\mathsf{d}(\cdot,\gamma_t) \le \frac{N}{\mathsf{d}(\cdot,\gamma_t)}\mathfrak{m}$$

Passing to the limit in the definition of $\underline{\Delta}$ we obtain

$$\underline{\Delta}\mathrm{b}^{\pm} \ge 0.$$

Use of the strong maximum principle

We have

$$\underline{\Delta}(b^+ + b^-) \geq 0,$$

the triangle inequality gives

$$b^+ + b^- \leq 0 \qquad \textit{on } X$$

and the fact that γ is a line ensures that

$$(b^+ + b^-)(\gamma_0) = 0$$

hence by the strong maximum principle it holds

$$b^+ + b^- \equiv 0$$

and in particular

$$\underline{\Delta} b^+ = \underline{\Delta} b^- = 0.$$

Put

$$b := b^+$$

PART III: Proof of the splitting in the non-smooth setting

- Busemann's function and harmonicity
- 'Gradient flow' of b and measure preservation
- 'Gradient flow' of b and isometries
- The quotient space and its isometric embedding
- Splitting
- Dimension reduction

The approach

The ‘standard’ theory of gradient flows in metric spaces yields no results for the gradient flow of b.

The approach

The ‘standard’ theory of gradient flows in metric spaces yields no results for the gradient flow of b.

A viable approach could be the one of using the notion of *Regular Lagrangian Flow* recently discussed by Ambrosio-Trevisan (generalization of the analogous notion on $\mathbb{R}^d$ introduced by Ambrosio in the context of Di Perna-Lions theory)

The approach

The ‘standard’ theory of gradient flows in metric spaces yields no results for the gradient flow of b.

A viable approach could be the one of using the notion of *Regular Lagrangian Flow* recently discussed by Ambrosio-Trevisan (generalization of the analogous notion on $\mathbb{R}^d$ introduced by Ambrosio in the context of Di Perna-Lions theory)

We shall instead use a different approach based on optimal transport and the peculiar properties of b

b as a Kantorovich potential

Lemma For any $t \in \mathbb{R}$ the function $t\mathrm{b}$ is c-concave and

$$(t\mathrm{b})^c = -t\mathrm{b} - \frac{t^2}{2}$$

Proof: basic application of the triangle inequality in conjunction with $\mathrm{b}^+ + \mathrm{b}^- = 0$.

b as a Kantorovich potential

Lemma For any $t \in \mathbb{R}$ the function $t\mathrm{b}$ is c-concave and

$$(t\mathrm{b})^c = -t\mathrm{b} - \frac{t^2}{2}$$

Proof: basic application of the triangle inequality in conjunction with $\mathrm{b}^+ + \mathrm{b}^- = 0$.

Note: If we were in $\mathbb{R}^d$:

- the thesis of the lemma would mean that b is affine
- $(x, y) \in \partial^c(t\mathrm{b})$ iff $y = x - t\nabla\mathrm{b}(x)$

Building the flow (F_t) as optimal maps

By the existence and uniqueness of optimal maps we deduce that for any $t \in \mathbb{R}$ and $\mathfrak{m}$-a.e. x there is a unique $F_{-t}(x)$ such that

$$(x, F_{-t}(x)) \in \partial^c(t\mathrm{b})$$

Building the flow (F_t) as optimal maps

By the existence and uniqueness of optimal maps we deduce that for any $t \in \mathbb{R}$ and $\mathfrak{m}$-a.e. x there is a unique $F_{-t}(x)$ such that

$$(x, F_{-t}(x)) \in \partial^c(t\mathrm{b})$$

The rigidity given by such optimality and the fact that $\underline{\Delta}\mathrm{b} = 0$ give, with little work, that

$$\begin{aligned} &(F_t)_*\mathfrak{m} = \mathfrak{m} \\ &F_t \circ F_s = F_{t+s} \qquad \mathfrak{m} - a.e. \end{aligned}$$

Building the flow (F_t) as optimal maps

By the existence and uniqueness of optimal maps we deduce that for any $t \in \mathbb{R}$ and $\mathfrak{m}$-a.e. x there is a unique $F_{-t}(x)$ such that

$$(x, F_{-t}(x)) \in \partial^c(t\mathrm{b})$$

The rigidity given by such optimality and the fact that $\underline{\Delta}\mathrm{b} = 0$ give, with little work, that

$$\begin{aligned} &(F_t)_*\mathfrak{m} = \mathfrak{m} \\ &F_t \circ F_s = F_{t+s} \qquad \mathfrak{m} - a.e. \end{aligned}$$

The fact that (F_t) solves "$\partial_t F_t = \nabla \mathrm{b} \circ F_t$" is read as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \,\mathrm{d}\mu_t = \int \langle \nabla f, \nabla \mathrm{b} \rangle \,\mathrm{d}\mu_t \qquad \forall t \in \mathbb{R}$$

for any $f \in W^{1,2}(X)$, $\mu \in \mathscr{P}_2(X)$ with bounded density and bounded support and $\mu_t := (F_t)_*\mu$.
This follows from 'horizontal and vertical derivatives'

PART III: Proof of the splitting in the non-smooth setting

- Busemann's function and harmonicity
- 'Gradient flow' of b and measure preservation
- 'Gradient flow' of b and isometries
- The quotient space and its isometric embedding
- Splitting
- Dimension reduction

‘Euler equation’ for b

Writing the Bochner inequality

$$\frac{1}{2}\int g|\mathrm{d}f|^2\,\mathrm{d}\mathfrak{m} \geq \int g\langle\nabla f, \nabla\Delta f\rangle\,\mathrm{d}\mathfrak{m}$$

for $\mathrm{b}+\varepsilon f$ in place of f we get, after simplification:

$$\int \Delta g\langle\nabla \mathrm{b}, \nabla f\rangle\,\mathrm{d}\mathfrak{m} = \int g\langle\nabla \mathrm{b}, \nabla\Delta f\rangle\,\mathrm{d}\mathfrak{m}$$

Energy preservation and isometries

With the same idea used in the smooth setting (and some intermediate regularization via the heat flow) one can show that

$$\int |\mathrm{d}(f \circ F_t)|^2 \,\mathrm{d}\mathfrak{m} = \int |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{m}$$

which coupled with $(F_t)_*\mathfrak{m} = \mathfrak{m}$ yields that

F_t is, up to redefinition on a negligible set, an isometry

PART III: Proof of the splitting in the non-smooth setting

- Busemann's function and harmonicity
- 'Gradient flow' of b and measure preservation
- 'Gradient flow' of b and isometries
- The quotient space and its isometric embedding
- Splitting
- Dimension reduction

The quotient space

Declare $x \sim y$ if $x = F_t(y)$ for some $t \in \mathbb{R}$, put $X' := X/\sim$ and define

$$\mathsf{d}'\big(\pi(x), \pi(y)\big) := \inf_{t \in \mathbb{R}} \mathsf{d}(x, F_t(y)) \qquad \forall x, y \in X$$

and

$$\mathfrak{m}'(E) := \mathfrak{m}(\pi^{-1}(E)) \cap \mathrm{b}^{-1}([0,1]) \qquad \forall E \subset X' \text{ Borel}$$

The quotient space

Declare $x \sim y$ if $x = F_t(y)$ for some $t \in \mathbb{R}$, put $X' := X/\sim$ and define

$$\mathsf{d}'\big(\pi(x), \pi(y)\big) := \inf_{t \in \mathbb{R}} \mathsf{d}(x, F_t(y)) \qquad \forall x, y \in X$$

and

$$\mathfrak{m}'(E) := \mathfrak{m}(\pi^{-1}(E)) \cap \mathrm{b}^{-1}([0,1]) \qquad \forall E \subset X' \text{ Borel}$$

The problem of the regularity of X'

A priori X' can be very irregular, but both to relate Sobolev functions on $X' \times \mathbb{R}$ to those on X and to deduce metric informations from this we would like it to be ‘well behaved’.

The problem of the regularity of X'

A priori X' can be very irregular, but both to relate Sobolev functions on $X' \times \mathbb{R}$ to those on X and to deduce metric informations from this we would like it to be 'well behaved'.
We shall prove that it isometrically embeds into X via the map $\iota : X' \to X$ given by

$$\iota(x') = x \qquad \text{if} \qquad \pi(x) = x' \quad \text{and} \quad \mathrm{b}(x) = 0.$$

This will grant that $(X', \mathrm{d}', \mathfrak{m}')$ is $\mathrm{RCD}(0, N)$.

The problem of the regularity of X'

A priori X' can be very irregular, but both to relate Sobolev functions on $X' \times \mathbb{R}$ to those on X and to deduce metric informations from this we would like it to be 'well behaved'.
We shall prove that it isometrically embeds into X via the map $\iota : X' \to X$ given by

$$\iota(x') = x \qquad \text{if} \qquad \pi(x) = x' \quad \text{and} \quad \mathrm{b}(x) = 0.$$

This will grant that $(X', \mathsf{d}', \mathfrak{m}')$ is $\mathrm{RCD}(0, N)$.
The fact that ι is an isometry is equivalent to the claim that for any $x, y \in X$ the minimum of

$$t \quad \mapsto \quad \mathsf{d}(x, F_t(y))$$

is attained at that t_0 such that $\mathrm{b}(x) = \mathrm{b}(F_{t_0}(y))$

How we could argue in the smooth case

Fix $x, y \in X$ be such that $t = 0$ is a minimizer of

$$t \mapsto \mathsf{d}^2\big(x, F_t(y)\big),$$

How we could argue in the smooth case

Fix $x, y \in X$ be such that $t = 0$ is a minimizer of

$$t \mapsto \mathsf{d}^2\big(x, F_t(y)\big),$$

let γ be a geodesic from x to y and notice that the triangle inequality and the fact that the F_t's are isometries grant that 0 $t = 0$ is a minimizer of

$$t \mapsto \mathsf{d}^2\big(x, F_t(\gamma_s)\big), \qquad \forall s \in [0, 1].$$

How we could argue in the smooth case

Fix $x, y \in X$ be such that $t = 0$ is a minimizer of

$$t \mapsto \mathsf{d}^2\big(x, F_t(y)\big),$$

let γ be a geodesic from x to y and notice that the triangle inequality and the fact that the F_t's are isometries grant that 0 $t = 0$ is a minimizer of

$$t \mapsto \mathsf{d}^2\big(x, F_t(\gamma_s)\big), \qquad \forall s \in [0, 1].$$

The Euler equation of the minimizer is

$$\langle \nabla \mathsf{d}^2(x, \cdot), \nabla \mathrm{b} \rangle(\gamma_s) = 0, \qquad \forall s \in [0, 1]$$

How we could argue in the smooth case

Fix $x, y \in X$ be such that $t = 0$ is a minimizer of

$$t \mapsto \mathsf{d}^2\big(x, F_t(y)\big),$$

let γ be a geodesic from x to y and notice that the triangle inequality and the fact that the F_t's are isometries grant that 0 $t = 0$ is a minimizer of

$$t \mapsto \mathsf{d}^2\big(x, F_t(\gamma_s)\big), \qquad \forall s \in [0, 1].$$

The Euler equation of the minimizer is

$$\langle \nabla \mathsf{d}^2(x, \cdot), \nabla \mathrm{b} \rangle(\gamma_s) = 0, \qquad \forall s \in [0, 1]$$

but $\nabla \mathsf{d}^2(x, \cdot)(\gamma_s) = 2s\gamma_s'$, thus the left hand side equals to

$$\frac{1}{2s} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{b}(\gamma_s)$$

showing that $\mathrm{b}(x) = \mathrm{b}(y)$

How to gain C^1 regularity in the RCD setting

Idea: lift the analysis from points to probability measures with bounded density.

How to gain C^1 regularity in the RCD setting

Idea: lift the analysis from points to probability measures with bounded density.
In practice, pick $\mu \in \mathscr{P}_2(X)$ with bounded density and bounded support, put $\mu_t := (F_t)_*\mu$ and minimize

$$t \qquad \mapsto \qquad \int \mathsf{d}^2(\cdot, x)\,\mathrm{d}\mu_t.$$

recalling that such map is C^1 with derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathsf{d}^2(\cdot, x)\,\mathrm{d}\mu_t = \int \langle \nabla \mathsf{d}^2(\cdot, x), \nabla \mathrm{b} \rangle\,\mathrm{d}\mu_t \qquad \forall t \in \mathbb{R}$$

How to gain C^1 regularity in the RCD setting

Idea: lift the analysis from points to probability measures with bounded density.
In practice, pick $\mu \in \mathscr{P}_2(X)$ with bounded density and bounded support, put $\mu_t := (F_t)_*\mu$ and minimize

$$t \qquad \mapsto \qquad \int \mathsf{d}^2(\cdot, x)\,\mathrm{d}\mu_t.$$

recalling that such map is C^1 with derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathsf{d}^2(\cdot, x)\,\mathrm{d}\mu_t = \int \langle \nabla \mathsf{d}^2(\cdot, x), \nabla \mathrm{b}\rangle\,\mathrm{d}\mu_t \qquad \forall t \in \mathbb{R}$$

Note: such C^1 regularity holds for general geodesics (μ_t) made of measures with bounded support and densities and generic $f \in W^{1,2}_{loc}(X)$ in place of $\mathsf{d}^2(\cdot, x)$

PART III: Proof of the splitting in the non-smooth setting

- Busemann's function and harmonicity
- 'Gradient flow' of b and measure preservation
- 'Gradient flow' of b and isometries
- The quotient space and its isometric embedding
- Splitting
- Dimension reduction

Mapping $X' \times \mathbb{R}$ to X and viceversa

Let $T : X' \times \mathbb{R} \to X$ be given by

$$T(x', t) := F_t(\iota(x'))$$

and $S : X \to X' \times \mathbb{R}$ by

$$S(x) := \big(\pi(x), \mathrm{b}(x)\big)$$

Mapping $X' \times \mathbb{R}$ to X and viceversa

Let $T : X' \times \mathbb{R} \to X$ be given by

$$T(x', t) := F_t(\iota(x'))$$

and $S : X \to X' \times \mathbb{R}$ by

$$S(x) := \big(\pi(x), \mathrm{b}(x)\big)$$

Easy to see:

- These are bijections
- The identity $(F_t)_* \mathfrak{m} = \mathfrak{m}$ grants that T, S are measure preserving
- The fact that F_t is an isometry yields that T, S are biLipschitz

Functions depending on 'one coordinate'

The isometric embedding of X' into X and the definition of $\mathfrak{m}'$ grant that

$$|\mathrm{d}(g \circ \pi)|_X = |\mathrm{d}g|_{X'} \circ \pi \qquad \mathfrak{m} - a.e.$$

for every $g \in W^{1,2}(X')$.

Functions depending on 'one coordinate'

The isometric embedding of X' into X and the definition of $\mathfrak{m}'$ grant that

$$|\mathrm{d}(g \circ \pi)|_X = |\mathrm{d}g|_{X'} \circ \pi \qquad \mathfrak{m} - a.e.$$

for every $g \in W^{1,2}(X')$.

Similarly, the fact that $\mathbb{R}$ embeds isometrically in X (via the straight line) and the fact that the F_t's are measure preserving ensure that

$$|\mathrm{d}(h \circ \mathrm{b})|_X = |\mathrm{d}h|_{\mathbb{R}} \circ \mathrm{b} \qquad \mathfrak{m} - a.e.$$

for every $h \in W^{1,2}(\mathbb{R})$.

The algebra generated by functions depending on ‘one coordinate’

Let $\mathcal{A} \subset W^{1,2}_{loc}(X' \times \mathbb{R})$ be the algebra generated by bounded Sobolev functions functions depending only on x' and on t.

We know that $\mathcal{A} \cap W^{1,2}(X' \times \mathbb{R})$ is dense in $W^{1,2}(X' \times \mathbb{R})$

Since T, S are biLipschitz and measure preserving $(\mathcal{A} \circ S) \cap W^{1,2}(X)$ is dense in $W^{1,2}(X)$

Conclusion of the argument: ‘splitting’ at the level of Sobolev functions

To conclude it is sufficient to prove that

$$|\mathrm{d}(f \circ S)|_X^2 = |\mathrm{d}f|_{X' \times \mathbb{R}}^2 \circ S \qquad \mathfrak{m} - a.e.$$

for every $f \in \mathcal{A}$

Conclusion of the argument: ‘splitting’ at the level of Sobolev functions

To conclude it is sufficient to prove that

$$|\mathrm{d}(f \circ S)|_X^2 = |\mathrm{d}f|^2_{X' \times \mathbb{R}} \circ S \qquad \mathfrak{m} - a.e.$$

for every $f \in \mathcal{A}$
We know that this is true for f depending only on x' or only on t, thus it remains to prove that

$$\langle \nabla g(x'), \nabla h(t) \rangle_{X' \times \mathbb{R}} = 0$$
$$\langle \nabla g(\pi(x)), \nabla h(\mathrm{b}(x)) \rangle_X = 0$$

Conclusion of the argument: 'splitting' at the level of Sobolev functions

To conclude it is sufficient to prove that

$$|\mathrm{d}(f \circ S)|_X^2 = |\mathrm{d}f|_{X' \times \mathbb{R}}^2 \circ S \qquad \mathfrak{m} - a.e.$$

for every $f \in \mathcal{A}$

We know that this is true for f depending only on x' or only on t, thus it remains to prove that

$$\langle \nabla g(x'), \nabla h(t) \rangle_{X' \times \mathbb{R}} = 0$$
$$\langle \nabla g(\pi(x)), \nabla h(\mathrm{b}(x)) \rangle_X = 0$$

- The first follows from the structure of Sobolev functions on $X' \times \mathbb{R}$
- The second from the chain rule and the 'horizontal and vertical derivates' which grants

$$\langle \nabla (g \circ \pi), \nabla \mathrm{b} \rangle_X = \lim_{t \to 0} \frac{g \circ \pi \circ F_t - g \circ \pi}{t} = 0$$

PART III: Proof of the splitting in the non-smooth setting

- Busemann's function and harmonicity
- 'Gradient flow' of b and measure preservation
- 'Gradient flow' of b and isometries
- The quotient space and its isometric embedding
- Splitting
- Dimension reduction

The case $N \in [1, 2)$

Theorem (Sturm '06) Let $(X, \mathsf{d}, \mathfrak{m})$ be a $CD(K, N)$ space.
Then the Hausdorff dimension of X is at most N.

The case $N \in [1, 2)$

Theorem (Sturm '06) Let $(X, \mathsf{d}, \mathfrak{m})$ be a $CD(K, N)$ space.
Then the Hausdorff dimension of X is at most N.

Proof: direct consequence of the Bishop-Gromov inequality

The case $N \in [1, 2)$

Theorem (Sturm '06) Let $(X, \mathrm{d}, \mathfrak{m})$ be a $CD(K, N)$ space.
Then the Hausdorff dimension of X is at most N.

Proof: direct consequence of the Bishop-Gromov inequality

For our application we notice that if X' is more than one point, it must contain a geodesic.
Thus the Hausdorff dimension of X should be at least 2.

The case $N \geq 2$

Lemma (Cavalletti, Sturm '12) Let $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ be essentially non-branching and such that $Y \times \mathbb{R}$ is $\mathsf{CD}(K, N)$ for $N \geq 2$. Then Y is a $\mathsf{CD}(K, N-1)$ space.

The case $N \geq 2$

Lemma (Cavalletti, Sturm '12) Let $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ be essentially non-branching and such that $Y \times \mathbb{R}$ is $\mathrm{CD}(K, N)$ for $N \geq 2$.
Then Y is a $\mathrm{CD}(K, N-1)$ space.

Proof: Pick $\mu_0, \mu_1 \in \mathscr{P}_2(Y)$ and use the convexity of the Renyi entropy for

$$\mu_0 \times \mathcal{L}^1|_{[0,1]} \qquad \text{and } \mu_1 \times (a^{-1}\mathcal{L}^1|_{[0,1]}).$$

Then optimize in $a > 0$.

Thank you

Some basic references

About the heat flow and the definition of RCD(K, ∞) spaces

Heat flow and calculus on metric measure spaces with Ricci curvature bounded below - the compact case, by Ambrosio, G., Savaré

in: Analysis and Numerics of Partial Differential Equations, 2013

Good overview to Sobolev functions on metric measure spaces, their relation with the curvature-dimension condition and the introduction of RCD(K, ∞) spaces.
Slightly outdated in some aspects: on RCD(K, ∞) spaces it is a priori required a convexity property of the Boltzamm entropy stronger than the one on CD(K, ∞) spaces. Thanks to some results of T. Rajala this turned out to be unnecessary, see in particular 'Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure' by Ambrosio, G., Mondino, Rajala, , Trans. Amer. Math. Soc., 2015.

Bochner inequality ($N = \infty$)

Heat flow on Alexandrov Spaces, by G., Kuwada, Ohta

in: CPAM, 2012

The setting is that of compact Alexandrov spaces, but Bochner inequality is proved as a consequence of only the EVI_K for the heat flow on probability measures, without relying on the Alexandrov structure. This is possibly the simplest presentation of the argument in the non-smooth setting and the proof directly generalizes to $\mathrm{RCD}(K, \infty)$ spaces.

Infinitesimally Hilbertian spaces, Laplacian comparison and splitting

An overview on the proof of the splitting theorem in non-smooth context, by G.

in: Analysis and Geometry in Metric Spaces, 2014

Self-contained presentation of the calculus tools in the non-smooth setting and their application to the proof of the splitting.
Note: for the maximum principle, uses the ‘nonlinear potential theory’ approach as discussed in the book by Bjorn-Bjorn instead of the argument discussed here.